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Abstract: Quantile regression (QR) models have been increasingly employed in many applied areas in economics. At the early stage, applications took place usually using cross-section data, but recent development has seen a surge of the use of quantile regression in both time-series and panel datasets. However, how to test for possible autocorrelation, especially in the context of time-series models, has been paid little attention. As a rule of thumb, one might attempt to apply the usual Breusch-Godfrey LM test to the residuals from the baseline quantile regression. In this paper, we demonstrate by Monte Carlo simulations that such an application of the LM test can result in potentially large size distortions, especially in either low or high quantiles. We then propose two correct tests (named the F-test and the QR-LM test) for autocorrelation in quantile models, which do not suffer from any size distortion. Monte Carlo simulation demonstrate that the two tests perform fairly well in finite samples, across either different quantiles or different underlying error distributions.

Keywords: Quantile regression, autocorrelation, LM test.

JEL classifications: C12, C22.

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1 Introduction

Since the pathbreaking work of Koenker and Bassett (1978), quantile regression models have been increasingly used in many applied areas in economics due to their flexibility to allow researchers to investigate the relationship between economic variables not only at the center but also over the entire conditional distribution of the dependent variable. In the early stage of the quantile regression literature, the main development in both theory and application has taken place mainly in the context of cross-section data. However, the application of quantile regression has subsequently moved into the areas of time-series as well as panel data. Some relevant papers are Koul and Mukherjee (1994), Koul and Saleh (1995), Koenker and Xiao (2004), Galvao (2009), Xiao (2009), Galvao et. al. (2008), Galvao et. al. (2009), Greenwood-Nimmo et. al. (2012), Cho et. al. (2012) in the time-series domain and Koenker (2004), Geraci and Bottai (2007), Abrevaya and Dahl (2008), Lamarche (2010), Galvao (2011) in the panel data setting.

When quantile regression is applied to time-series data models, it is particularly important to check for any sign of autocorrelation because, as correctly pointed out by Koenker (2005, pp. 128), the typical IID (independent and identically distributed errors) condition is usually imposed on the regression error terms in both mean-regression and quantile regression. If the error terms are serially correlated in quantile models, the effect is the same as in any mean models; that is, employing the usual variance-covariance matrix for inference becomes invalid.

In the quantile regression literature, there has been some effort to investigate the asymptotic properties and the least absolute deviations (LAD) estimator, which is a special case of regression quantile, and to derive the correct asymptotic variance-covariance matrix of the LAD estimator in the presence of autocorrelation. For example, see Davis and Dunsmuir (1997) and Weiss (1990, 1991). However, little attention has been paid to the issue of how to test for autocorrelation in quantile regression models. The only significant exception is the work of Furno (2000) where the performance of the conventional Breusch-Godfrey LM test applied to LAD regression models has been studied by Monte Carlo simulations.

In the absence of any theoretical prescription, one might attempt as a rule of thumb to apply the usual LM test to the residuals from the baseline quantile regression. In this paper, we demonstrate by Monte Carlo simulations that such an application of the LM test can result in potentially large size distortions, especailly in either low or high quantiles. For example, when the error distribution is the standard normal, the empirical rejection rates for the null hypothesis of no autocorrelation are around 30% for both 0.05 and 0.95 quantiles when the nominal size is 5%. Given that quantile models and usually regarded as a supplementary tool for OLS-type mean regression and therefore mainly used to investigate either low and high quantiles, having such severe size distortions can be a serious problem. After documenting the phenomenon of spurious autocorrelation induced by the usual LM test, we then propose two correct tests (named the F-test and the QR-LM test) for autocorrelation in quantile models, which do not suffer from any size distortion. Monte Carlo simulation demonstrate that the two tests perform fairly well in finite samples, across either different quantiles or different underlying error distributions.

This paper is organized as follows. In section 2, we demonstrate the effect of blindly applying the LM test to quantile regression models. Then, two alternative tests for autocorrelation are proposed and their finite sample performance is studied in Section 3. A concluding remark is provided in Section 4.

2 Spurious Autocorrelation in Quantile Models

We consider following quantile linear regression model:

$$y_t = h(x_t, \beta_\theta) + \epsilon_{\theta t}, \qquad t = 1, \cdots, T \tag{1}$$

where x_t is a $k \times 1$ vector of explanatory variables whose first column is one, T denotes the number of observations for estimation, $\theta \in (0, 1)$ is a pre-specified quantile level, and β_θ is the vector of unknown quantile parameter to be estimated. Note that the quantile model in (1) is potentially nonlinear and the vector (x_t) of explanatory variables can include some lags of the dependent variable.[1] We will assume that there is no endogenous variable in x_t.

Since autocorrelation is a common phenomenon for time series data, it might be possible that the error terms $\epsilon_{\theta t}$ are serially correlated. In the presence of autocorrelated errors, quantile estimates become less inefficient and more importantly any inference based on the conventional variance-covariance matrix becomes invalid. Therefore, the importance of checking for autocorrelation in quantile models is just the same as in OLS-type mean regression models. Nevertheless, no formal investigation into this important issue has been carried out so far. One possible reason is that researchers might have assumed that the conventional LM test typically used in OLS-type regression models could be also applicable in quantile regression models. The Objective of this paper is to show that this rule-of-thumb approach can result in potentially significant size distortions (i.e. spurious autocorrelation) and to propose some valid testing procedures for checking for autocorrelation in quantile models.

In this section, we first demonstrate the strange phenomenon of spurious autocorrelation caused by the naive application of the LM test. The standard LM test is based on the quantile residuals $(e_{\theta t})$ obtained from the following quantile regression:

$$y_t = h(x_t, \hat{\beta}_\theta) + e_{\theta t},$$

where $\hat{\beta}_\theta$ is the quantile regression estimator. Once $e_{\theta t}$ are obtained, the auxiliary regression of $e_{\theta t}$ on x_t and $e_{\theta t-1}, e_{\theta t-2}, \cdots, e_{\theta t-p}$ is carried out as follows:

$$e_{\theta t} = x_t' \tilde{\beta} + \hat{\rho}_1 e_{\theta, t-1} + \hat{\rho}_2 e_{\theta, t-2} + \cdots + \hat{\rho}_p e_{\theta, t-p} + \hat{v}_t, \tag{2}$$

where $\hat{v}_t$'s are the residuals. The LM statistic is given by $T \times R^2$ where R^2 is the R-square from the auxiliary regression in (2), and the LM statistic is compared with an appropriate critical value from the Chi-square distribution with p degrees of freedom.

Suppose that the error terms in (1) are not serially correlated; i.e. $E(\epsilon_{\theta t}\epsilon_{\theta s}) = 0$ for $t \neq s$. Then, any rejection of the null of no autocorrelation larger than a pre-specified significance level indicates the phenomenon of spurious regression. To show the performance of the LM test explained above, we design our data generation process (DGP) and simulations as follows. We assume a linear model and include only one scalar exogenous variable (z_t) in the model so that the DGP is given by:

$$y_t = \beta_{0\theta} + \beta_{1\theta} z_t + \epsilon_{\theta t}, \tag{3}$$

where z_t are independently generated from the standard normal distribution. To be able to focus on the size property of the LM test, the error terms $\epsilon_{\theta t}$ are also independently generated from a distribution. We consider four different distributions to generate the error terms, which are the standard normal distribution $N(0, 1)$, and the student t-distributions with 5 and 1 degrees of

[1] In such a case, the obvious stationarity condition should be imposed.

freedom (DF) [$t(5)$, $t(1)$] to represent thin, moderately heavy and extremely heavy tailed symmetric distributions, respectively, and lastly the lognormal distribution with $\mu = 1, \sigma = 0.4$ (centered at zero by subtracting the mean value) denoted by $LN(1, 0.4)$ to represent an asymmetric distribution.

Once z_t and $\epsilon_{\theta t}$ are generated as described above, the dependent variable y_t are generated through (3) with $\beta_{0\theta} = 1$ and $\beta_{1\theta} = 1$ and with various sample sizes $T = 50, 300, 500, 1000, 3000, 5000$. For our initial simulations, we choose $p = 2$ for the auxiliary regression in (2) although we study the effect of varying the value of p later. The rejection probability (i.e. the empirical size) of the LM test is calculated through 1,000 replications and the nominal size is fixed at 5% in all of simulations. Table 1 shows the empirical sizes of the LM test for various quantile indexes $\theta = 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95$. We also include the empirical sizes of the LM test for LS regression as well for the purpose of comparison. Table 1 consists of four panels, each of which corresponds to each of the four error distributions ($N(0,1), t(5), t(1)$, and $LN(1, 0.4)$). In each panel, sample sizes are in the row while each column represents a different quantile index (θ).

We first consider the $N(0, 1)$ case displayed in the top panel in Table 1. When the LM test is carried out for LS regression, the empirical sizes are fairly close to the nominal size 5%, regardless of the sample size, which is expected and well-known. However, when the LM test is implemented for quantile models, we have very different results. To begin with, when based on median (or LAD) regression with $\theta = 0.5$, the rejection frequencies, although slightly over-sized, are more or less close to the nominal size. But, as θ moves towards either low or high quantiles, the size distortion increases symmetrically around $\theta = 0.5$. For examples, the rejection probability is about 20% for $\theta = 0.1, 0.9$ while it increases even further to around 30% for $\theta = 0.05, 0.95$. It is interesting to note that size distortions do not seem to decrease or to vanish as the sample size increases, even up to $T = 5,000$. The second panel of Table 1 shows the rejection frequencies of the LM test for quantile models when the error term is generated by the moderately fat-tailed distribution $t(5)$. The results are quite similar to the $N(0, 1)$ case, except that size distortions become more pronounced at extreme quantiles like $\theta = 0.05, 0.95$ where the empirical sizes are in the range of 35% to 45%.

The case that the error term is generated from the extremely fat-tailed $t(1)$ distribution is shown in the third panel of Table 1. Different from the previous two cases, the most of the rejection frequencies are much smaller and depend on the sample size. When $T = 50$, the empirical sizes are either close to or above the 5% nominal size, but they start to decrease as the sample size increases. The decreasing trend continues even after $T = 5,000$ so that we have added the separate case with $T = 10,000$. In such a large sample, the LM test is severely under-size, especially for LS regression; the rejection probability is just 1.2%. The decreasing trend of empirical sizes also applies to quantile models as well although it is less pronounced than LS regression. The finding is consistent with Weiss (1990) where the possible inconsistency of the LM test is pointed out when the error term in auxiliary regression does not satisfy the first moment condition.

The bottom panel of Table 1 shows the results when the error term is generated from the asymmetric lognormal distribution $LN(1, 0.4)$. The results are in sharp contrast to all of the previous cases in that the empirical sizes are no longer symmetric around the median $\theta = 0.5$. The rejection probability increases moderately as θ moves to lower quantiles (e.g. it is about 14% when $\theta = 0.05$; an increase by 6% from the median value of about 8%) whereas it increases sharply as θ moves towards the high end of the distribution (e.g. the rejection probability is about 50% when $\theta = 0.95$; an increase by 42% from the median value). All of the previous results on the rejection probability of the LM test presented in Table 1 is again graphically displayed in Figures 1-4, each of which corresponds to each of the four error distributions. The figures visually confirm our previous discussion.

Thus far, we have fixed the number of lags used in the auxiliary regression at two ($p = 2$) in all of simulations. One might be curious on the effect of using different values for p. We take the results with $\theta = 0.9$ from Table 1 as the baseline case and attempt to use other values of p; specifically $p = 1, 2, 4$, and 12, which are typically used in applications depending on the frequency (monthly, quarterly, or yearly) of data. Table 2 shows the results of this new simulations again for each of the four error distributions.

It is well known that using different values of p does not have any effect on the size property of the LM test in LS regression. This is confirmed in the first 4 columns in Table 2. The rejection probability does not vary with p when based on LS regression. The only exception occurs when $p = 12$ and $T = 50$. For examples, the empirical size is 1.5% for the $N(0, 1)$ case with $p = 12$ and $T = 50$. This must be due to the lack in the degree of freedom in the auxiliary regression for the LM test because there are too many parameters to be estimated relative the sample size. An interesting observation is that the severe under-size problem previously discovered for the Cauchy error distribution $t(1)$ appears to become lessened by using large values of p.

The invariance property of the LM test with respect to the value of p for LS regression, however, breaks down for quantile regression as clearly shown in columns 5-8 in Table 2. A common pattern over all of the four error distributions is that the rejection probability tends to decrease as p increases. For examples, the rejection probability with $T = 500$ in the case of the $t(5)$ error distribution is given by 28.6%, 21,8%, 17.5% and 10.7% for $p = 1, 2, 4$, and 12, respectively. Therefore, the size distortion problem in the LM test used in quantile regression models can be lessened by employing a large value of p. However, not only it will not eliminate the problem completely, but also it is likely to cause a reduction in power of the LM test. In light of this observation, we propose two correct test for autocorrelation that can be used in quantile regression models in the next section.

3 Alternative Tests for Autocorrelation in Quantile Models

In this section, we propose two alternative tests for autocorrelation that can be used in quantile regression models without any size distortion. The first test we propose is the usual F-test for the null hypothesis that all of the parameters for the lagged residuals in the auxiliary regression are jointly zero. The second test is a quantile extension of the autocorrelation test proposed in Weiss (1990) where the baseline regression is carried out by the LAD estimation method.

3.1 The F-test

Considering the auxiliary regression in (2), the null hypothesis implied by no autocorrelation is

$$H_0 : \rho_1 = \rho_2 = ...\rho_p = 0.$$

The test statistic (denoted by W) for the F-test for this null hypothesis is given by

$$W = \frac{(\sum_{t=1}^{T} \tilde{v}_t^2 - \sum_{t=1}^{T} \hat{v}_t^2)/p}{\sum_{t=1}^{T} \hat{v}_t^2/(T - p - k)},$$

where $\hat{v}_t$ is the residuals from the unrestricted auxiliary regression in (2), $\tilde{v}_t$ is the residuals from the restricted auxiliary regression where the null hypothesis is imposed, and the statistic W is asymptotically distributed as the Chi-square distribution with p degrees of freedom.

For the purpose of comparison, we employ the identical simulation setup as in the previous section. The simulation results for the proposed F-test in quantile models are shown in Table 3 along with its counterpart in LS regression. It is clearly demonstrated that the F-test does not show any sign of size distortion for all regular error distribution $(N(0, 1), t(5)$ and $LN(1, 0.4)$. For the Cauchy distribution, we still observe the same problem that the empirical size tend to converge towards zero as the sample size increases. In fact, the numbers reported in Table 3 seem to indicate that the empirical size may converge to 1.2% in both LS and quantile regression models. As before, the effect of changing p is also simulated and the results shown in Table 4 confirms that the desired invariance property of the F-test with respect to the number of lagged residuals in the auxiliary regression.

3.2 The QR-LM Test

For simplicity, let us assume that the function in (1) is linear; $h(x_t, \beta_\theta) = x_t'\beta_\theta$. Assuming some regularity conditions, Kim and White (2003) proved that

$$T^{-1/2} \sum_{t=1}^{T} x_t \psi_\theta(e_{\theta t}) = o_p(1), \tag{4}$$

where $e_{\theta t} = y_t - x_t'\hat{\beta}_\theta$, $\hat{\beta}_\theta$ is the $\theta^{th}-$quantile estimator, and $\psi_\theta(z) = \theta - 1_{[z \leq 0]}$. The result in (4) can be considered as the orthogonality condition between the regressor x_t and the transformed residuals $\psi_\theta(e_{\theta t})$. Given such an orthogonality condition, we can use an argument in Weiss (1990, pp137) to consider the following auxiliary regression:

$$\psi_\theta(e_{\theta t}) = x_t'\bar{\beta} + \tilde{\rho}_1 e_{\theta,t-1} + \tilde{\rho}_2 e_{\theta,t-2} + \cdots + \tilde{\rho}_p e_{\theta,t-p} + \tilde{\eta}_t. \tag{5}$$

The modified auxiliary regression in (5) is the same as the one in (2) except that the dependent variable is not the residual itself $e_{\theta t}$, but the transformed residual $\psi_\theta(e_{\theta t})$. The LM statistic is given by $T \times R^2$ where R^2 is the R-square from the modified auxiliary regression in (5). The new LM test is called as the QR-LM test. Under the null hypothesis that $\rho_1 = \rho_2 = ...\rho_p = 0$, it can be shown that the QR-LM statistic follows asymptotically the Chi-square distribution with p degrees of freedom.[2]

Using the same simulation setup as before, the performance of the QR-LM test is simulated and the results are reported in Table 5. As shown in Table 5, the QR-LM test performs quite well in terms of size, regardless of the choice of different quantile indexes or different error distributions again except the Cauchy distribution. Unlike the previous F-test, the QR-LM test tends to be over-sized for both low and high quantiles for the Cauchy distribution. The desired invariance property with respect to p for the QR-LM test is demonstrated in Table 6. It can be concluded

[2] It can be shown that the QR-LM statistic ($T \times R^2$) can be numerically same as the following statistic (denoted as QLM):

$$QLM = S(\theta)' I(\theta)^{-1} S(\theta),$$

where

$$\begin{aligned}
S(\tilde{r}) &= \begin{pmatrix} \sum_{t=1}^{T} x_t \psi_\theta(e_{\theta t}) \\ \sum_{t=1}^{T} \tilde{e}_t \psi_\theta(e_{\theta t}) \end{pmatrix}, \\
I(\tilde{r}) &= \begin{pmatrix} \sum_{t=1}^{T} \psi_\theta(e_{\theta t})^2 x_t' x_t & \sum_{t=1}^{T} \psi_\theta(e_{\theta t})^2 x_t' \tilde{e}_t \\ \sum_{t=1}^{T} \psi_\theta(e_{\theta t})^2 \tilde{e}_t' x_t & \sum_{t=1}^{T} \psi_\theta(e_{\theta t})^2 \tilde{e}_t' \tilde{e}_t \end{pmatrix}, \\
\tilde{e}_t &= (e_{\theta,t-1}, \cdots, e_{\theta,t-p})'.
\end{aligned}$$

that the QR-LM test is comparable to the F-test in terms of size for regular error distributions. On the other hand, if one is concerned with some extreme error distribution like the Cauchy, the F-test can be better because it becomes just conservative in such a case whereas the QR-LM test becomes over-sized.

By now we could conclude that F-test and QR-LM test for autocorrelation of the error term in quantile regression are more plausible than conventional LM test under null none autocorrelation hypothesis. In the following part, the comparison between F-test and QR-LM test while error term is autocorrelated will be investigated. Simulation is constructed identically as previous, except for generating error from AR(1), that is ,

$$e_t = \rho e_{t-1} + \eta_t \tag{6}$$

where η_t is also generated from previous four distributions ($N(0,1)$, $t(5)$, $t(1)$,$LN(1,0.4)$), and the initial value of e_t, e_1, is set to be zero. For each sample size T, $\{e_t, t = 2, 3, \cdots, T + 50\}$ are generated by equation 6 and the first 50 observations, $e_1, e_2, \cdots, e_{50}$, will be dropped to get rid of the impact of initial value.

Table 7 and 8 display the rejection probabilities of F-test ($p = 2$) while $\rho = 0.4$ and $\rho = 0.8$, respectively. In Table 7, while $\rho = 0.4$, rejection frequencies for four distributions are quite similar, where rejection frequencies are about 50% to 60% and more than 90% for these four distributions while sample size is $T = 50$ and $T = 100$, respectively. As sample size rises to $T = 200$ or more, test demonstrates to accept null hypothesis with probability 99% or 100%. Compare to Table 7, test results while $\rho = 0.8$ in Table 8 exhibit extremely high rejection frequencies even as sample size $T = 50$. No big difference between autocorrelation tests by OLS and quantile regression at different quantiles are found.

For the rejection probability of QR-LM test while $\rho = 0.4$ in Table 9, although it can be seen that the trend of rejection frequency is getting close to 100% according to increasing sample sizes , the power is lower than corresponding results of F-test in Table 7, especially for $T(1)$ distribution, whose rejection frequencies are rather lower than the other three distributions as $T = 300$. Rejection frequencies of QR-LM test are higher while $\rho = 0.8$ as shown in Table 10 than that while $\rho = 0.4$ in Table 9, however, still lower than the corresponding F-test results. It is noted that the highest rejection frequencies are located at the middle quantile, and with quantile decreasing or increasing from middle to either tail, rejection probabilities are getting smaller and smaller(Only exception is the case of $t(1)$ distribution while $\rho = 0.4$). However, no matter the detailed location of the distribution is, no rejection frequencies are higher than LM test results by OLS regression.

We could conclude that, while underlying error is autocorrelated, F-test is not impacted by quantiles and performs more powerful than QR-LM test, and the power of QR-LM is higher at the middle than at the tails but still lower than conventional LM test by OLS regression. We recommend F-test for autocorrelation in quantile regression models.

4 Test Application For Fama-French Three-Factor Model

Fama and French (1996) proposed three-factor model to eliminate anomalies in CAPM model, which are excess return on a market portfolio($r_M - r_f$), difference between the return on a portfolio of small stocks and that of large stocks (SMB, small minus big), and difference between the return on a portfolio of low book-to-market stocks and that of high book-to-market stocks (HML, high minus low). The model is written as :

$$(r_A - r_f) = \alpha_A + \beta_A(r_M - r_f) + s_A SMB + h_A HML + e \tag{7}$$

where $r_A - r_f$ is $T \times 1$ the excess return vector of an asset. α_A, β_A, s_A, and h_A are intercept, and coefficients for above three factors, respectively. e is the error term.

Allen et.al (2009) applied quantile regression in Fama-French three-factor model to reveal differences between these factors at different quantile, by using a data set of 30 returns of Dow Jones Index stocks and factors downloaded from the data library of Professor Ken French's website over the period from 2002 to 2009. We would not argue or analyse the behaviour of Fara-French three-factor model, but simply implement our autocorrelation tests for the error term generated from quantile regression to discover the existence of serial correlation in the data, and therefore whether the quantile estimates are efficient.

We set up a data set of the monthly return of Microsoft Corporation (MSFT) in Nasdaq (r_A) from yahoo finance website, three factors ($r_M - r_f$,SMB,HML) collected from Professor Ken French's website and 3-month treasury bill as risk free rate (r_f) downloaded Federal Reserve Bank of st.Louis site, which period is from July, 2006 to June, 2012. Quantile regression estimates, statistics and its p-value of conventional LM test, F-test and QR-LM test are displayed in Table 11, along with OLS results as a comparison.

It is noted that the test results perform differently according to different lag length p. In order to decide the proper lag length for autocorrelation test, we compute Akaike information criterion (AIC) and Bayesian information criterion (BIC) with different lag length, which are commonly used as fit measures for model selection besides adjusted R^2. For the residuals of quantile regression at θ, e_θ, AIC and BIC are calculated as (see Greene 5ed pp.160 and Hayashi textbook pp.396)

$$AIC = \log(e_\theta' e_\theta / \tilde{T}) + 2(k+p)/\tilde{T}$$
$$BIC = \log(e_\theta' e_\theta / \tilde{T}) + (k+p)\log(\tilde{T})/\tilde{T}$$

Where $\tilde{T} = T - p$, k is the number of regressors in (7) and equals to 4, including intercept, and then $k+p$ stands for the number of these regressors and up to p^{th} lagged residuals of (7), which appear in the auxiliary model as independent variables. We assume the influence would not last more than one year, i.e. the biggest p equals to 12 and then compute the AIC and BIC for $p = 1, 2, \cdots, 12$. The appropriate lag length , denoted as $\hat{p}_{AIC}$ and $\hat{p}_{BIC}$, would be determined by the corresponding p of the minimum value of AIC and BIC respectively. Our calculation concludes that, for F-test, $\hat{p}_{AIC} = 1$ as quantile $\theta = 0.05, 0.10, 0.25, 0.50, 0.75$ and $\hat{p}_{AIC} = 4$ as quantile $\theta = 0.90, 0.95$, while $\hat{p}_{BIC} = 1$ for all quantiles. Similarly for QR-LM test, at most quantiles $\hat{p}_{AIC} = 1$, while at all quantiles we considered here $\hat{p}_{BIC} = 1$. Based on our calculation, we would like to determine the appropriate lag length is $\hat{p}_{AIC} = \hat{p}_{BIC} = 1$ for the auxiliary regression of QR-LM test and list the results while $p = 1$ and $p = 4$, which is used as a comparison, in Table 11.

The top panel in Table 11 shows OLS and quantile regression estimates which verifies the differences of the coefficients across different quantiles. For the LM test and F-test by OLS regression in the two case, $p = 1, 4$ below, there is no significant rejection of null hypothesis based on nominal size of 5%. In second panel of $p = 1$ case, conventional LM test accepts the null at $\theta = 0.75, 0.90$ only, however, F-test never rejects and QR-LM rejects at $\theta = 0.90, 0.95$ only. The conventional LM test obviously rejects null more frequently, compared to F-test and QR-LM test, which also could be observed in the third panel($p = 4$). Corresponding to $\hat{p}_{AIC} = 4$ at quantile $\theta = 0.90, 0.95$, the test results are displayed in the last two columns in the last panel of Table 11, which indicates that at the extremely higher quantiles, conventional LM test strongly reject null hypothesis, while F-test rejects at $\theta = 0.95$ and accepts at $\theta = 0.90$ but not strongly and QR-LM test accepts null. Overall, autocorrelation tests application to Fama-French model performs similar results to our simulation that conventional LM test results in potentially large size distortions, however, F-test and QR-LM test perform fairly well, of which F-test is better.

5 Conclusion

Although quantile regression has been increasingly employed in the context of time-series data, no formal investigation of how to test for autocorrelation in quantile regression models has been carried out. In this paper we demonstrate the phenomenon of spurious autocorrelation that occurs when the conventional LM test is blindly applied in quantile regression models. We find that the size distortion problem is more pronounced in either low or high quantiles than the middle quantiles. Given that quantile regression is mainly used for either low or high quantiles, the finding indicates that researchers can conclude with high probability that the quantile errors are serially correlated when they are in fact not. We then propose two alternative testing procedures (the F-test and the QR-LM test) that are free of any size distortion. Based on the comparison of their size properties, we recommend to use the F-test instead of the QR-LM test. As a by-product of our investigation, we have discovered that when the errors are drawn from extremely fat-tailed distributions, neither the conventional LM test nor our proposed tests can work properly. We leave the issue of finding a fix in such a circumstance for our future work.

References

[1] Abrevaya, J., Dahl, C.M., 2008. The effects of birth inputs on birthweight: evidence from quantile estimation on panel data. Journal of Business and Economic Statistics 26, 379-397.

[2] Allen D.E., Singh A.K., Powell R., 2009. Asset Pricing, the Fama-French factor model and the implications of quantile regression analysis, working paper.

[3] Bloomfield,P., Steiger, W., 1983. Least absolute deviations: theory, applications, and algorithms. Boston: Birkhauser.

[4] Breusch, T.S., 1978. Testing for autocorrelation in dynamic linear models. Australian Economic Papers 17, 334-355.

[5] Cho, J.S, Kim, T.-H., Shin, Y., 2012, Quantile cointegration using autoregressive distributed-lag modeling approach, discussion paper.

[6] Davis, R. A., Dunsmuir, W.T.M., 1997. Least absolute deviation estimation for regression with ARMA errors. Journal of Theoretical Probability 10, 481-497.

[7] Fama, E.F., French K.R., 1996. Multifactor Explanations of Asset Pricing Anomalies. The Journal of Finance 1,55-83.

[8] Furno, M., 2000. LM tests in the presence of non-normal error distributions, Econometric Theory 16, 249-261.

[9] Galvao, A.F., 2009. Unit root quantile autoregression testing using covariates. Journal of Econometrics 152, 165-178.

[10] Galvao, A.F., 2011. Quantile regression for dynamic panel data with fixed effects. Journal of Econometrics 164, 142-157.

[11] Galvao, A.F., Montes-Rojas, G., Olmo, J., 2008. Threshold quantile autoregressive models, discussion paper.

[12] Galvao, A.F., Montes-Rojas, G., Park, S.Y., 2009. Quantile autoregressive distributed lag model with an application to house price returns, discussion paper.

[13] Geraci, M., Bottai, M., 2007. Quantile regression for longitudinal data using the asymmetric Laplace distribution. Biostatistics 8, 140-154.

[14] Godfrey, L.G., 1978. Testing for higher order serial correlation in regression equations when the regressors include lagged dependent variables. Econometrica 47, 1303-1310.

[15] Greenwood-Nimmo, M., Kim, T-H., Shin, Y., Treeck, T., 2012. Fundamental asymmetries in US monetary policymaking: evidence from a nonlinear autoregressive distributed lag quantile regression model, discussion paper.

[16] Hasan, M. N., Koenker, R., 1997. Robust rank tests of the unit root hypothesis. Econometrica 65, 133-161.

[17] Knight, K., 1989. Limit theory for autoregressive-parameter estimates in an infinite-variance random walk. The Canadian Journal of Statistics 17, 261-278.

[18] Kim, T.-H., White,H., 2003. Estimation, inference, and specification testing for possibly misspecified quantile regression. Advances in Econometrics 17, 107-132.

[19] Koenker, R., 2004. Quantile regression for longitudinal data. Journal of Multivariate Analysis 91, 74-89.

[20] Koenker, R., 2005. Quantile regression. Cambridge University Press.

[21] Koenker, R., Bassett,G., 1978. Regression quantiles, Econometrica 46, 33-49

[22] Koenker, R., Xiao, Z., 2004. Unit root quantile autoregression inference. Journal of the American Statistical Association 99, 775-787.

[23] Koul, H., Mukherjee,K., 1994. Regression quantiles and related processes under long range dependent errors. Journal of Multivariate Analysis 51, 318-337.

[24] Koul, H., Saleh A., 1995. Autoregression quantiles and related rank score processes. Annals of Statistics 23, 670-689.

[25] Lamarche, C. 2010. Robust penalized quantile regression estimation and inference for panel data, Journal of Econometrics 157, 396-408.

[26] Weiss, A.A., 1990. Least absolute error estimation in the presence of serial correlation, Journal of Econometrics 44, 127-158.

[27] Weiss, A.A., 1991. Estimating nonlinear dynamic models using least absolute error estimation, Econometric Theory 7, 46-68.

[28] Xiao, Z., 2009. Quantile cointegrating regression, Journal of Econometrics 150, 248–260.

Table 1. Rejection frequencies (%) of the conventional LM Test ($p = 2$)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	5.7	29.5	19.6	10.2	7.7	10.8	20.5	29.9
300	5.1	34.1	19.0	9.9	7.0	8.5	18.9	31.1
500	5.2	36.4	23.2	10.9	7.4	10.3	21.6	33.6
1,000	4.5	33.2	19.2	9.9	6.3	9.4	19.3	33.1
3,000	4.6	31.2	21.8	8.1	6.7	8.7	18.7	31.1
5,000	5.6	30.2	20.5	10.0	8.3	10.1	21.3	32.6
				$t(5)$				
50	4.2	35.8	21.7	7.6	6.1	7.3	20.1	37.6
300	4.3	37.9	21.9	7.9	6.3	8.2	21.6	38.4
500	5.0	40.7	23.1	9.2	7.2	9.8	21.8	44.6
1,000	5.3	38.7	20.4	8.7	7.7	10.3	21.5	39.4
3,000	5.4	41.1	22.6	9.3	7.4	9.0	22.0	42.3
5,000	5.2	40.8	23.4	9.7	6.9	10.3	22.3	43.3
				$t(1)$				
50	4.0	36.1	11.4	4.9	5.3	5.2	13.0	34.8
300	3.1	16.5	6.4	5.5	5.6	5.6	6.1	15.2
500	3.4	12.1	5.3	5.0	5.4	5.5	5.6	13.4
1,000	3.3	9.7	6.2	5.8	5.5	6.0	6.0	10.0
3,000	1.8	4.7	3.1	3.3	3.5	3.4	3.6	5.3
5,000	1.9	6.0	4.1	4.3	4.2	4.5	4.1	4.8
10,000	1.2	4.0	2.8	3.0	3.2	3.2	3.2	3.9
				$LN(1, 0.4)$				
50	5.0	14.9	12.4	8.4	7.3	10.4	31.8	49.6
300	4.5	14.3	11.0	7.9	7.2	9.9	32.3	51.3
500	4.8	15.6	12.3	8.7	7.3	12.4	36.7	53.7
1,000	4.9	15.5	11.8	8.6	7.2	11.2	33.9	55.8
3,000	4.8	13.3	11.4	9.3	7.9	10.6	34.0	53.8
5,000	5.4	12.5	11.8	9.0	8.2	12.4	36.5	52.0

Table 2. Rejection frequencies (%) of the conventional LM-Test

	OLS				Quantile regression ($\theta = 0.9$)			
T	$p=1$	$p=2$	$p=4$	$p=12$	$p=1$	$p=2$	$p=4$	$p=12$
				$N(0,1)$				
50	5.4	5.7	5.3	1.5	28.5	20.5	14.0	2.2
300	5.5	5.1	4.8	3.9	24.8	18.9	15.7	8.6
500	4.8	5.2	5.8	4.6	28.2	21.6	16.4	9.6
1,000	5.2	4.5	3.9	4.9	24.7	19.3	14.6	11.4
3,000	4.5	4.6	4.2	4.3	24.9	18.7	14.4	9.1
5,000	4.7	5.6	6.3	3.9	27.4	21.3	17.0	10.1
				$t(5)$				
50	4.5	4.2	3.3	0.7	28.1	20.1	11.9	2.4
300	4.8	4.3	4.3	3.5	29.5	21.6	16.2	8.4
500	6.1	5.0	5.3	4.9	28.6	21.8	17.5	10.7
1,000	5.3	5.3	4.7	4.5	27.7	21.5	15.0	11.4
3,000	3.7	5.4	4.7	3.9	27.7	21.9	16.9	11.2
5,000	4.4	5.2	3.7	4.4	29.0	22.3	16.2	12.0
				$t(1)$				
50	2.8	4.0	4.3	4.9	16.0	13.0	10.2	3.9
300	2.4	3.1	4.9	7.6	8.7	6.1	6.0	7.6
500	1.9	3.4	3.4	6.0	7.8	5.6	4.4	5.9
1,000	2.5	3.3	4.6	8.5	7.9	6.0	5.6	9.0
3,000	1.5	1.8	3.2	5.3	6.4	3.6	4.1	5.6
5,000	1.2	1.9	2.2	3.8	5.7	4.1	2.5	3.9
10,000	1.1	1.3	1.7	4.4	6.2	3.3	2.0	4.5
				$LN(1, 0.4)$				
50	4.8	5.0	4.8	1.1	41.3	31.8	22.5	4.3
300	4.7	4.5	5.1	2.4	40.1	32.3	25.7	16.2
500	4.6	4.8	4.3	4.4	43.2	36.7	27.9	18.8
1,000	4.9	4.9	4.9	4.9	43.1	33.9	26.6	18.3
3,000	5.0	4.8	4.7	4.2	40.1	34.0	27.0	17.6
5,000	4.0	5.4	5.3	4.2	40.4	36.5	30.0	18.6

Table 3. Rejection frequencies (%) of the F-test ($p = 2$)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0, 1)$				
50	5.4	5.3	5.8	5.2	5.2	5.3	4.5	5.3
300	4.9	4.5	4.8	5.0	4.8	4.7	5.3	4.9
500	5.2	5.3	5.4	5.2	5.3	5.1	5.3	5.1
1,000	4.5	4.8	4.8	4.4	4.5	4.5	4.6	4.8
3,000	4.6	4.6	4.6	4.6	4.5	4.6	4.6	4.7
5,000	5.6	5.8	5.7	5.7	5.5	5.6	5.4	5.5
				$t(5)$				
50	4.0	3.5	4.3	3.6	3.9	4.0	4.1	3.7
300	4.3	4.7	4.5	4.4	4.4	4.3	4.7	4.7
500	5.0	5.0	5.1	4.5	4.8	5.2	4.9	5.2
1,000	5.3	5.4	5.3	5.3	5.3	5.3	5.1	4.8
3,000	5.4	5.3	5.2	5.1	5.5	5.3	5.3	5.3
5,000	5.2	5.2	5.2	5.0	5.2	5.2	5.1	5.3
				$t(1)$				
50	3.0	3.0	2.8	3.0	2.9	3.0	3.5	3.7
300	3.0	3.0	3.0	3.1	3.1	3.1	3.1	3.2
500	3.3	3.1	3.2	3.2	3.1	3.1	3.2	3.3
1,000	3.3	3.2	3.3	3.3	3.3	3.3	3.3	3.2
3,000	1.7	1.6	1.6	1.7	1.6	1.7	1.6	1.8
5,000	1.9	1.9	1.9	1.9	1.9	1.9	1.9	2.0
10,000	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
				$LN(1, 0.4)$				
50	5.0	4.5	4.7	4.5	4.7	4.7	4.7	4.0
300	4.3	4.3	4.2	4.2	4.6	4.5	4.3	4.4
500	4.8	5.0	5.0	4.8	4.7	4.9	5.2	4.5
1,000	4.9	5.0	5.3	5.2	5.0	4.9	5.1	4.9
3,000	4.8	5.1	4.9	5.0	4.9	5.0	4.9	4.7
5,000	5.4	5.4	5.4	5.4	5.3	5.2	5.2	5.2

Table 4. Rejection frequencies (%) of the F-Test

T	OLS $p=1$	OLS $p=2$	OLS $p=4$	OLS $p=12$	Quantile regression ($\theta=0.9$) $p=1$	$p=2$	$p=4$	$p=12$
				$N(0,1)$				
50	5.0	5.4	5.1	2.2	4.8	4.5	4.5	1.9
300	5.4	4.9	4.8	3.9	5.2	5.3	5.1	4.0
500	4.8	5.2	5.8	4.6	5.0	5.3	5.7	4.5
1,000	5.2	4.5	3.9	4.9	5.2	4.6	3.7	5.1
3,000	4.5	4.6	4.2	4.3	4.4	4.6	4.5	4.4
5,000	4.7	5.6	6.3	3.9	4.8	5.4	6.3	3.8
				$t(5)$				
50	3.8	4.0	3.2	2.4	3.8	4.1	3.4	2.2
300	4.8	4.3	4.3	3.6	4.8	4.7	4.4	3.5
500	6.1	5.0	5.2	5.0	6.5	4.9	5.2	4.6
1,000	5.3	5.3	4.6	4.6	5.1	5.1	4.4	4.5
3,000	3.7	5.4	4.7	3.9	3.8	5.3	4.7	4.0
5,000	4.4	5.2	3.7	4.4	4.4	5.1	3.8	4.4
				$t(1)$				
50	2.3	3.0	3.2	3.5	2.3	3.5	3.3	3.3
300	2.3	3.0	4.8	7.1	2.3	3.1	4.9	7.4
500	1.8	3.3	3.3	5.4	1.8	3.2	3.4	5.5
1,000	2.5	3.3	4.6	8.4	2.4	3.3	4.6	8.4
3,000	1.4	1.7	3.1	5.2	1.4	1.6	3.1	5.2
5,000	1.2	1.9	2.2	3.8	1.2	1.9	2.2	3.8
10,000	1.1	1.3	1.7	4.4	1.1	1.3	1.7	4.5
				$LN(1,0.4)$				
50	4.3	5.0	4.6	2.0	3.8	4.7	3.5	2.0
300	4.6	4.3	5.1	2.6	4.9	4.3	4.9	3.0
500	4.4	4.8	4.3	4.5	4.7	5.2	4.3	4.2
1,000	4.8	4.9	4.9	4.9	4.9	5.1	4.6	4.6
3,000	5.0	4.8	4.7	4.2	5.1	4.9	5.1	4.5
5,000	4.0	5.4	5.3	4.2	3.9	5.2	5.3	4.0

Table 5. Rejection frequencies (%) of the QR-LM test

T	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$			
50	6.2	6.0	6.5	7.1	6.5	6.1	5.9
300	5.3	5.6	5.3	4.9	5.1	5.5	3.8
500	5.9	4.7	5.5	4.5	7.0	5.1	5.6
1,000	5.9	5.1	5.1	6.0	4.7	5.4	5.4
3,000	6.6	4.3	4.2	3.6	4.4	4.8	5.7
5,000	4.8	4.3	5.3	5.1	4.9	4.9	4.9
				$t(5)$			
50	7.6	5.3	5.7	5.2	6.3	6.7	7.6
300	4.7	5.3	4.3	4.8	6.0	4.2	5.0
500	4.0	5.5	5.9	4.6	5.4	4.1	4.4
1,000	5.2	4.4	5.2	5.3	5.3	5.7	6.4
3,000	5.5	4.8	5.0	4.7	5.1	4.8	6.1
5,000	5.9	4.6	5.9	4.9	5.6	5.2	4.0
				$t(1)$			
50	8.3	11.0	5.0	2.3	5.2	13.3	10.6
300	9.6	10.5	4.3	1.5	4.1	12.4	11.3
500	10.6	11.9	4.6	2.7	4.7	11.3	10.5
1,000	10.8	12.5	3.0	2.2	4.6	10.8	9.6
3,000	10.1	11.5	3.7	2.0	4.4	11.4	8.9
5,000	10.8	11.2	3.2	1.4	4.3	10.2	8.0
10,000	8.4	10.6	3.4	1.2	4.3	15.1	11.1
				$LN(1,0.4)$			
50	8.1	7.0	6.6	7.3	6.6	5.0	4.2
300	6.3	6.4	5.3	5.3	5.1	4.9	3.5
500	6.0	4.9	5.2	5.5	6.6	5.1	5.7
1,000	5.2	6.3	4.1	5.3	4.2	4.9	5.1
3,000	6.4	4.8	4.1	3.8	4.6	4.7	6.4
5,000	4.9	4.3	5.4	5.7	4.9	5.2	5.0

Table 6. Rejection frequencies (%) of the QR-LM Test

	Quantile regression ($\theta = 0.9$)			
T	$p = 1$	$p = 2$	$p = 4$	$p = 12$
		$N(0,1)$		
50	6.7	6.1	5.1	2.6
300	4.4	5.5	5.2	4.1
500	5.0	5.1	5.2	3.3
1,000	4.4	5.4	5.4	5.2
3,000	6.3	4.8	4.4	4.4
5,000	4.0	4.9	5.5	5.8
		$t(5)$		
50	7.0	6.7	5.6	3.1
300	5.0	4.2	2.9	3.9
500	4.3	4.1	5.5	4.6
1,000	6.3	5.7	5.8	4.7
3,000	4.6	4.8	6.9	4.8
5,000	5.6	5.2	4.8	4.9
		$t(1)$		
50	8.5	13.3	9.8	7.8
300	8.5	12.4	10.4	10.5
500	7.9	11.3	9.4	11.3
1,000	7.4	10.8	9.1	9.8
3,000	6.8	11.4	10.6	10.8
5,000	7.9	10.2	10.1	10.9
10,000	10.3	15.1	13.2	13
		$LN(1, 0.4)$		
50	5.5	5.0	4.4	2.1
300	4.5	4.9	5.3	4.1
500	5.0	5.1	4.8	4.2
1,000	5.2	4.9	5.2	5.1
3,000	5.6	4.7	4.8	4.9
5,000	4.6	5.2	5.8	5.1

Table 7. Rejection frequencies(%) of F-Test ($p = 2$, $\rho = 0.4$)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	58.3	56.5	57.4	58.7	58.2	58.2	56.3	56.4
100	94.2	93.1	94.2	93.8	93.9	93.5	93.1	93.1
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(5)$				
50	61.0	55.8	58.0	60.0	60.3	59.9	58.4	55.3
100	94.0	91.7	93.1	93.4	93.9	93.3	93.6	92.1
200	99.7	99.7	99.7	99.7	99.7	99.7	99.7	99.7
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(1)$				
50	61.3	50.3	60.2	64.1	64.5	64.1	59.3	50.7
100	97.2	94.7	96.9	97.3	97.4	97.4	97.0	94.4
200	99.4	99.5	99.4	99.4	99.4	99.4	99.4	99.2
300	99.9	99.9	99.9	99.9	99.9	99.9	99.9	99.9
				$LN(1,0.4)$				
50	56.9	57.3	58.1	57.9	57.3	55.6	53.6	50.4
100	93.9	93.9	93.8	93.7	93.9	93.1	92.4	89.4
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 8. Rejection frequencies(%) of F-Test($p = 2$,$\rho = 0.8$)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	99.8	99.6	99.8	99.8	99.8	99.9	99.7	99.5
100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(5)$				
50	99.5	99.6	99.4	99.5	99.4	99.5	99.4	99.3
100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(1)$				
50	99.0	98.0	98.7	98.9	98.9	99.0	98.9	97.8
100	99.8	99.8	99.8	99.8	99.8	99.8	99.8	99.8
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$LN(1,0.4)$				
50	99.9	99.7	99.9	99.8	99.8	99.9	99.7	99.4
100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 9. Rejection frequencies(%) of QR-LM Test ($p = 2$, $\rho = 0.4$)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	60.2	13.1	19.8	36.0	40.2	34.0	18.4	16.1
100	94.3	28.9	43.2	70.5	77.1	70.6	47.5	32.2
200	100.0	62.9	82.2	97.5	99.0	96.2	80.7	61.1
300	100.0	83.5	95.8	100.0	100.0	99.9	95.6	81.9
				$t(5)$				
50	62.5	12.6	20.5	39.8	51.8	39.6	21.2	13.6
100	94.3	29.1	44.6	76.7	87.2	76.8	47.8	31.2
200	99.7	51.8	80.1	98.6	99.7	98.3	80.6	51.5
300	100.0	71.4	94.2	99.8	100.0	99.7	93.9	71.0
				$t(1)$				
50	64.1	32.2	43.2	58.6	61.5	58.4	43.1	29.6
100	97.3	56.4	68.4	79.6	84.6	80.2	67.5	54.3
200	99.4	73.8	79.6	86.6	89.5	86.8	80.2	74.4
300	99.9	79.8	84.7	90.0	91.4	89.9	85.7	82.6
				$LN(1,0.4)$				
50	58.3	8.8	16.3	44.0	51.1	36.4	21.3	17.2
100	94.3	33.1	63.4	88.3	89.0	68.5	41.5	30.4
200	100.0	86.7	98.1	99.8	99.7	96.2	69.7	47.7
300	100.0	99.2	100.0	100.0	100.0	99.7	86.5	60.7

Table 10. Rejection frequencies(%) of QR-LM Test ($p = 2$, $\rho = 0.8$)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	99.8	39.7	65.2	94.5	97.5	92.7	66.4	42.3
100	100.0	89.9	98.7	99.9	100.0	100.0	98.8	89.7
200	100.0	99.9	100.0	100.0	100.0	100.0	100.0	99.8
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(5)$				
50	99.5	40.0	64.8	94.0	98.3	93.6	66.6	40.7
100	100.0	86.5	98.0	100.0	100.0	99.9	98.3	87.2
200	100.0	99.4	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(1)$				
50	99.0	35.8	58.0	87.8	98.2	87.7	57.7	38.0
100	99.8	73.2	84.5	95.1	99.1	95.4	85.8	72.2
200	100.0	85.7	91.5	96.9	99.8	96.5	90.6	86.0
300	100.0	90.2	92.8	97.2	100.0	97.6	93.0	90.2
				$LN(1,0.4)$				
50	99.9	37.9	66.0	97.2	98.3	93.9	66.4	45.0
100	100.0	92.4	99.7	100.0	100.0	100.0	97.5	85.1
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	98.9
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 11. Quantile regression application for Fama-French Model(72 observations)

	OLS	$\theta = 0.05$	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 0.90$	$\theta = 0.95$
				quantile estimate of coefficients				
β	1.1148	1.0209	1.2454	1.1325	0.9428	1.1361	0.9949	0.9155
s	-0.2575	-0.8273	-0.7291	-0.7272	-0.3897	0.0559	0.1677	0.6085
h	-0.1605	0.3592	0.1245	-0.0410	0.2436	-0.2948	-0.3536	-0.7212
intercpt	0.3770	-6.2329	-4.9197	-3.1117	0.0297	3.9406	5.3457	8.7135
				$p = 1$				
LM stat.	1.0913	9.1255	6.4235	3.9166	5.3538	1.9412	3.2296	13.1499
p-value	0.2962	0.0025	0.0113	0.0478	0.0207	0.1635	0.0723	0.0003
F stat.	1.0084	0.8619	1.3569	1.1463	0.6847	0.9445	0.6572	0.3884
p-value	0.3190	0.3566	0.2483	0.2882	0.4110	0.3347	0.4205	0.5353
QR-LM stat.	Na	3.3787	0.9190	3.3380	1.4393	1.8434	5.0179	4.6803
p-value	Na	0.0660	0.3377	0.0677	0.2302	0.1745	0.0251	0.0305
				$p = 4$				
LM stat.	8.3804	12.9819	10.8495	10.5307	10.3244	9.2772	11.2645	20.6249
p-value	0.0786	0.0114	0.0283	0.0324	0.0353	0.0545	0.0237	0.0004
F stat	2.1003	1.3715	1.5176	2.0459	1.5542	2.1347	2.3658	2.6770
p-value	0.0919	0.2545	0.2085	0.0993	0.1982	0.0875	0.0629	0.0402
QR-LM stat	Na	11.1700	7.1447	6.9767	4.8389	6.9404	7.5786	7.5865
p-value	Na	0.0247	0.1284	0.1371	0.3042	0.1391	0.1083	0.1080

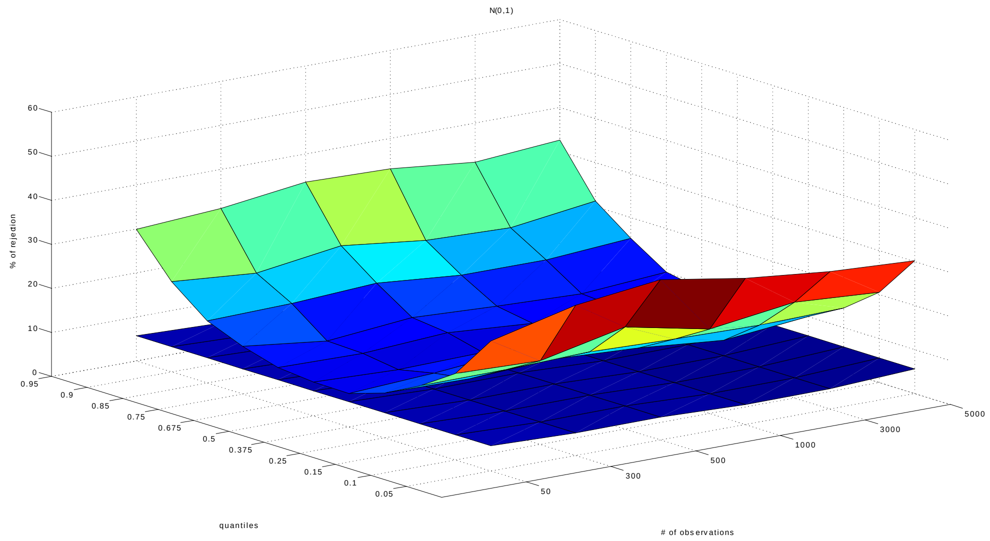

Figure 1. Rejection frequencies (%) of the conventional LM Test ($p = 2$) when errors are drawn from $N(0, 1)$.

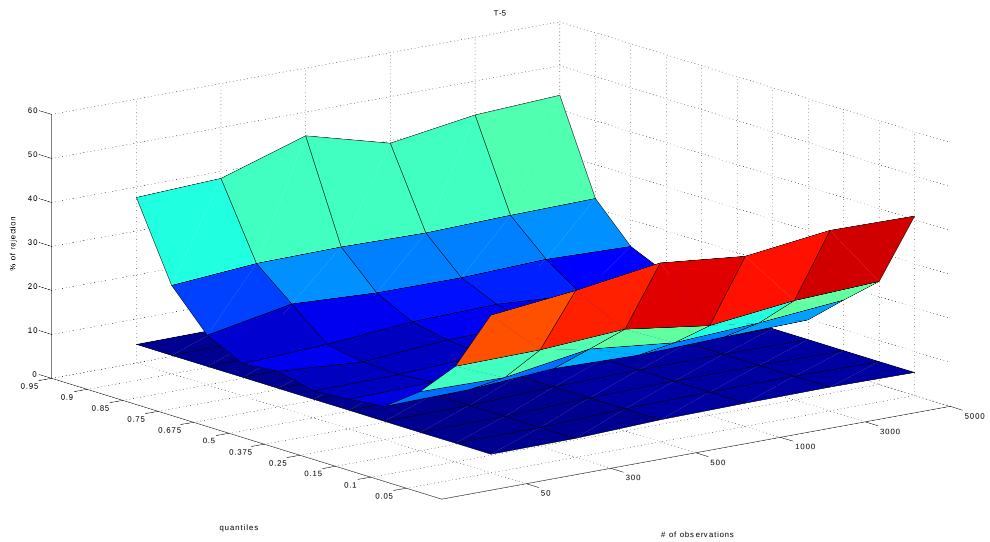

Figure 2. Rejection frequencies (%) of the conventional LM Test ($p = 2$) when errors are drawn from $t(5)$.

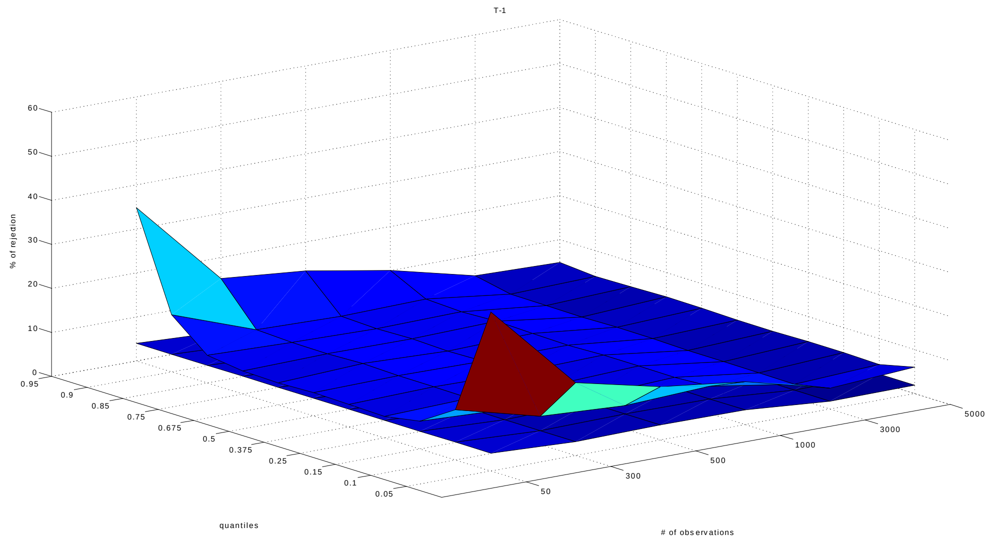

Figure 3. Rejection frequencies (%) of the conventional LM Test ($p = 2$) when errors are drawn from $t(1)$.

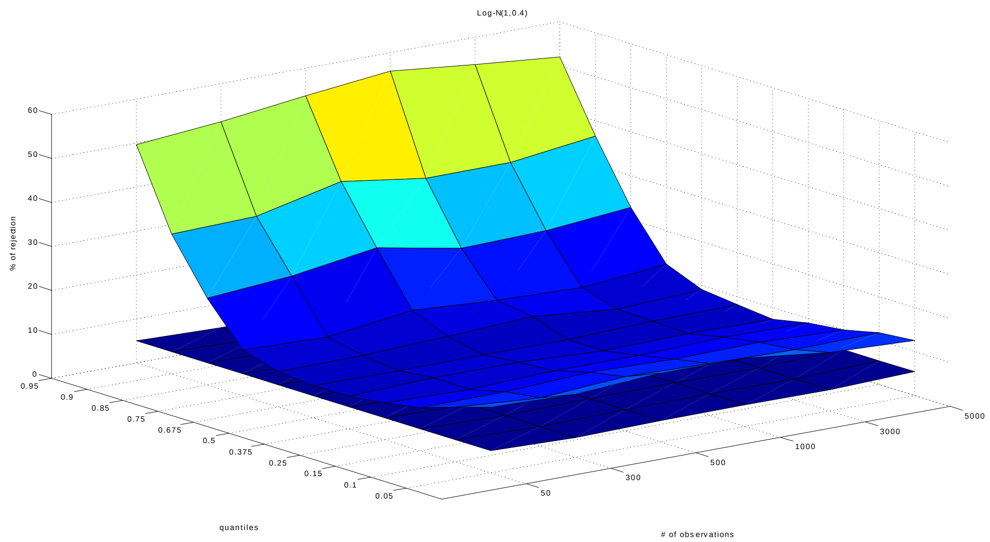

Figure 4. Rejection frequencies (%) of the conventional LM Test ($p = 2$) when errors are drawn from $LN(1, 0.4)$.