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Abstract

We start from deriving several option pricing formulas of which the prices of underlying asset follow three different commonly used heavy-tailed distribution functions: Generalized Pareto Distribution, Generalized Logistic Distribution, and Generalized Extreme Value Distribution. The derived option pricing formulas differently characterize the probabilities of extreme events in financial markets, so that these can be used to overcome the conventional "normal" market assumption by capturing the negative skewness and/or excess kurtosis of financial asset returns. We then compare how option prices behave depending on the assumed distributions with their scale and shape parameters.

1 INTRODUCTION

The extreme events in a financial market such as the 1987 stock market crash, the Asian crisis, the September 11th attacks, and the credit crunch of 2008 that was triggered by the bankruptcy of Lehman Brothers have one thing in general: they were much more common than conventional theory anticipates. They were unexpected, therefore not capable of being managed by the financial practitioner and policymakers who now consider them as anticipated events. Extracting the risk-neutral probability density function from key asset is one way of estimating the markets' view of extreme and rare events in financial market. In comparison with estimating markets' statistics such as implied volatility, deriving risk-neutral density function, which is a full probability density function, provides much richer information of markets' expectation, allowing viewers to see how risk-neutral expectation of market participants has evolved over time. Hence, producing the implied risk-neutral density function is worthwhile for the following reasons:

i. It provides the central tendency of markets' expectation from which the center of density function shifts toward higher or lower values.

ii. It provides the dispersion of market participants' expectation from which the density function is less concentrated or more widely spread out.

iii. If the density function is skewed to the right or the left, it means that investors are more concerned about changes in one direction than the other.

iv. If the tail of density function has become substantial, it indicates that the probability of large change has increased.

Identifying the risk-neutral density function can be broadly classified into parametric and non-parametric techniques. Parametric method is choosing a distribution family and then trying to estimate the parameters for these distributions that are consistent with the observed price. Our approach is to select some heavy-tailed distributions, then to derive an option pricing formula of these distributions and to eventually identify the parameters of these distributions that minimize the sum of squared of observed option price.

The Black-Scholes model is a way of estimating the value of options in a financial market. Despite of its popularity and practicality, it seems imperfect because systematic errors occur between observed prices and theoretical prices given by the model. These systematic errors are caused by the assumption of the Black-Scholes model that the distribution of the underlying price is a log-normal distribution. As an alternative to the assumption of Black-Scholes model, and also to capture the view that extreme events in a financial market have significant probability, the necessity of heavy-tailed distribution has been raised.

In this paper, we focus on deriving the heavy-tailed based option pricing model explicitly. We start with generalized logistic distribution(GLO) which is also called shifted log logistic distribution. It has three parameters with location, scale and shape parameter. Unlike the log normal distribution, GLO distribution is a fat tailed distribution exhibiting large skewness and kurtosis with the addition of shape parameter.

2 THE GLO OPTION PRICING MODEL

2.1 Generalized Logistic Distribution

The generalized logistic distribution has distribution function:

$$\mathbf{F_{GLO}}(x;\mu,\sigma,\xi) = \begin{cases} \frac{1}{1+\left(1+\frac{\xi(x-\mu)}{\sigma}\right)^{-1/\xi}} & \text{for} \quad \xi \neq 0 \\ \frac{1}{1+e^{-\frac{x-\mu}{\sigma}}} & \text{for} \quad \xi = 0 \end{cases} \tag{1}$$

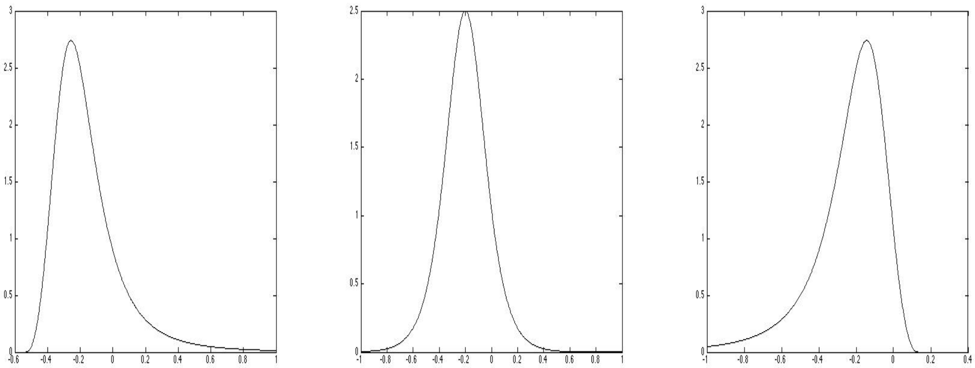
for $1+\xi(x-\mu)/\sigma > 0$, where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ the scale parameter and $\xi \in \mathbb{R}$ the shape parameter. When $\xi > 0$, it is valid for $x \in [\mu - \frac{\sigma}{\xi}, +\infty)$ and while $\xi < 0$, it is valid for $x \in (-\infty, \mu - \frac{\sigma}{\xi}]$. Note that when $\xi \to 0$, the generalized logistic distribution (GLO) reduces to logistic distribution (LO) and when $\xi = 1$, it is the same as the generalized pareto distribution (GPD) with shape parameter $\xi = 1$. Figure (1) illustrates GLO distributions with same location($\mu = -0.2$), scale parameter($\sigma = 0.1$) and different shape parameters($\xi = 0.3$, 0, -0.3). It can be easily seen that GLO distribution is skewed to right when shape parameter is positive and skewed to left when shape parameter is negative. As stated above, GLO distribution reduces to logistic distribution when shape parameter is zero. Logistic distribution is a symmetric distribution exhibiting zero skewness.

2.2 European Call with GLO returns

The arbitrage-free European call option price formula is

$$C_t(K) = e^{-r(T-t)} \cdot \int_K^\infty \big(S_T - K\big) f(S_T) dS_T \tag{2}$$

Here, $f(S_T)$ is a risk-neutral density function of the underlying price at maturity. We assume that simple negative returns is assumed to follow generalized logistic distribution(GLO) with the positive shape parameter($\xi > 0$).



(a) $\xi > 0$ (b) $\xi = 0$ (c) $\xi < 0$

Figure 1: GLO distributions with $\mu = -0.2$, $\sigma = 0.1$

$$L_T = -R_T = -\frac{S_T - S_t}{S_t} = 1 - \frac{S_T}{S_t} \tag{3}$$

$$f_{GLO}(L_T) = \frac{\big(1 + (\xi/\sigma)(L_T - \mu)\big)^{-1/\xi - 1}}{\sigma\big(1 + \big(1 + (\xi/\sigma)(L_T - \mu)\big)^{-1/\xi}\big)^2} \quad \text{with} \quad \xi > 0 \tag{4}$$

When shape parameter is positive, the variable of generalized logistic distribution is bounded by its support function($x \in [\mu - \sigma/\xi, +\infty)$). Since we assume simple negative returns follow generalized logistic distribution, underlying price at maturity is bounded as $-\infty < S_T \leq S_t(1 - \mu + \sigma/\xi)$. In context of the option pricing model, the positive shape parameter means that there are great probability mass on the extreme loss and density function of underlying price is right skewed.

Note that the relationship between the density function for L_T and S_T is given by the general formula.

$$f(S_T) = f(L_T)\left|\frac{\partial L_T}{\partial S_T}\right| = \frac{1}{S_T} f L_T) \tag{5}$$

With the support function of generalized logistic distribution, the upper limit of call option price in Equation (2) becomes $S_t(1 - \mu + \sigma/\xi)$ and the density function of underlying price at maturity is substituted by Equation (5).

Hence, the GLO based option pricing model is

$$C_t(K) = e^{-r(T-t)} \cdot \int_K^{S_t(1-\mu+\frac{\sigma}{\xi})} \big(S_T - K\big) \frac{1}{S_t} f_{GLO}(L_T) dS_T \tag{6}$$

To solve the GLO based call option price, applying change of variable technique to the GLO density function,

$$y = 1 + \frac{\xi(L_T - \mu)}{\sigma} = 1 + \frac{\xi}{\sigma}\left(1 - \frac{S_T}{S_t} - \mu\right) > 0 \tag{7}$$

then the underlying price S_T and dS_T can be written in terms of y as follows:

$$S_T = S_t\Big(1 - \mu - \frac{\sigma}{\xi}(y-1)\Big) \quad \text{and} \quad dS_T = -S_t\frac{\sigma}{\xi}dy \tag{8}$$

Also, the upper limit of integration for the call option becomes zero and the lower limit of integration becomes as follows:

$$H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_t} - \mu\right) \tag{9}$$

Lastly, using the new limit of integration and substituting for S_T and dS_T defined in Equation (8), we have GLO call option equation:

$$C_t(K) = e^{-r(T-t)} \cdot \int_H^0 \left(S_t\left(1 - \mu - \frac{\sigma(y-1)}{\xi}\right) - K\right) \frac{-y^{-(1+\xi)/\xi}}{\xi(1+y^{-1/\xi})^2} dy \tag{10}$$

$$\text{where} \quad H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_t} - \mu\right)$$

We separate the above equation into two terms that the payoff function with y variable and without y variable.

$$\begin{aligned} C_t(K) &= e^{-r(T-t)} \cdot \int_H^0 \left(S_t \left(1 - \mu + \frac{\sigma}{\xi} \right) - K \right) \frac{-y^{-(1+\xi)/\xi}}{\xi \left(1 + y^{-1/\xi}\right)^2} dy \\ & \quad - e^{-r(T-t)} \cdot \int_H^0 \left(S_t \frac{\sigma}{\xi} \right) \frac{-y^{-1/\xi}}{\xi (1 + y^{-1/\xi})^2} dy \end{aligned} \tag{11}$$

Consider the first integral term.

$$\Psi_1 = -\frac{1}{\xi} \int_H^0 \frac{y^{-(1+\xi)/\xi}}{\left(1 + y^{-1/\xi}\right)^2} dy \tag{12}$$

The integral Ψ_1 can be solved easily by applying change of variable technique, $y^{-1/\xi} = t$, to the above equation.

$$\begin{aligned} \Psi_1 &= \int_{H^{-1/\xi}}^{\infty} \frac{1}{(1+t)^2} dt \\ &= -\frac{1}{1+t} \bigg|_{H^{-1/\xi}}^{\infty} = \frac{1}{1 + H^{-1/\xi}} \end{aligned} \tag{13}$$

Consider the second integral term.

$$\Psi_2 = -\frac{1}{\xi} \int_H^0 \frac{y^{-1/\xi}}{\left(1 + y^{-1/\xi}\right)^2} dy \tag{14}$$

As same as the first integral term, Ψ_1, apply change of variable, $y^{-1/\xi} = t$, to the Equation (14).

$$\Psi_2 = \int_{H^{-1/\xi}}^{\infty} \frac{t^{-1/\xi}}{(1+t)^2} dt \tag{15}$$

The integral Ψ_2 can be solved using the modification of beta function and incomplete beta function.

$$B(a,b) = \int_0^1 t^{a-1}\big(1-t\big)^{b-1} dt = \int_0^\infty \frac{u^{a-1}}{\big(1+u\big)^{a+b-2}} du \tag{16}$$

$$B(x,a,b) = \int_0^x t^{a-1}\big(1-t\big)^{b-1} dt = \int_0^u \frac{u^{a-1}}{\big(1+u\big)^{a+b-2}} du \tag{17}$$

$$\text{where} \quad u = \frac{t}{1-t}, \quad 0 < x < 1, \quad 0 < u < \infty$$

The integral Ψ_2 can be rearranged as follow:

$$\Psi_2 = \int_0^\infty \frac{t^{-1/\xi}}{(1+t)^2} dt - \int_0^{H^{-1/\xi}} \frac{t^{-1/\xi}}{(1+t)^2} dt \tag{18}$$

The first term of Ψ_2 can be solved with the modification of beta function, and the second term of Ψ_2 can be solved with the modification of incomplete beta function.

$$\Psi_2 = B(1-\xi, 1+\xi) - B\left(\frac{H^{-1/\xi}}{1+H^{-1/\xi}}, 1-\xi, 1+\xi\right) \tag{19}$$

Finally, we solved all integral terms, then we have the closed form of call option

$$\begin{aligned} C_t(K) = S_t \cdot e^{-r(T-t)} \cdot & \left(\left(1-\mu+\frac{\sigma}{\xi}\right)\left(\frac{1}{1+H^{-1/\xi}}\right)\right. \\ & \left. - \frac{\sigma}{\xi}\Big(B(1-\xi,1+\xi) - B(t_1, 1-\xi, 1+\xi)\Big)\right) \\ - K \cdot e^{-rT} \cdot & \left(\frac{1}{1+H^{-1/\xi}}\right) \end{aligned} \tag{20}$$

$$\text{where} \quad t_1 = \frac{H^{-1/\xi}}{1+H^{-1/\xi}}$$

2.3 European Put with GLO returns

Start with the arbitrage-free put option formula which is

$$P_t(K) = e^{-r(T-t)} \int_0^K (K - S_T) f(S_T) dS_T \tag{21}$$

Unlike call option formula, put option will be exercised at maturity when underlying price below the given price. With the assumption that negative returns said to follow generalized logistic distribution, the density function of underlying price at maturity is substituted with $f_{GLO}(L_T)/S_t$. Hence, GLO based put option pricing model is

$$P_t(K) = e^{-r(T-t)} \int_0^K (K - S_T) \frac{1}{S_t} f_{GLO}(L_T) dS_T \tag{22}$$

To solve Equation (22), apply change of variable, $y = 1 + \frac{\xi(L_T - \mu)}{\sigma}$, in same way as call option pricing model. The upper limit of integration of put option becomes $H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_t} - \mu\right)$ and the lower limit of integration becomes $h = 1 + \frac{\xi}{\sigma}(1 - \mu)$.

$$P_t(K) = e^{-r(T-t)} \cdot \int_h^H \left(K - S_t\left(1 - \mu - \frac{\sigma(y-1)}{\xi}\right)\right) \frac{-y^{-(1+\xi)/\xi}}{\xi(1 + y^{-1/\xi})^2} dy \tag{23}$$

$$\text{where} \quad H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_t} - \mu\right) \quad \text{and} \quad h = 1 + \frac{\xi}{\sigma}(1 - \mu)$$

Rearrange the above Equation (23) as two integral terms.

$$\begin{aligned} P_t(K) &= e^{-r(T-t)} \cdot \int_h^H \left(K - S_t\left(1 - \mu + \frac{\sigma}{\xi}\right)\right) \frac{-y^{-(1+\xi)/\xi}}{\xi\left(1 + y^{-1/\xi}\right)^2} dy \\ &\quad + e^{-r(T-t)} \cdot \int_h^H \left(S_t \frac{\sigma}{\xi}\right) \frac{-y^{-1/\xi}}{\xi(1 + y^{-1/\xi})^2} dy \end{aligned} \tag{24}$$

The first integral term can be solved by applying change of variable, $y^{-1/\xi} = t$, and the second term can be solved with the modification of incomplete beta function.

$$
\begin{aligned}
\Psi_3 &= \int_h^H \frac{-y^{-(1+\xi)/\xi}}{\xi\left(1+y^{-1/\xi}\right)^2}dy = \int_{h^{-1/\xi}}^{H^{-1/\xi}} \frac{1}{(1+t)^2}dt \\
&= \frac{1}{1+h^{-1/\xi}} - \frac{1}{1+H^{-1/\xi}}
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\Psi_4 &= \int_h^H \frac{-y^{-1/\xi}}{\xi\left(1+y^{-1/\xi}\right)^2}dy = \int_{h^{-1/\xi}}^{H^{-1/\xi}} \frac{t^{-\xi}}{(1+t)^2}dt \\
&= \int_0^{H^{-1/\xi}} \frac{t^{-\xi}}{(1+t)^2}dt - \int_0^{h^{-1/\xi}} \frac{t^{-\xi}}{(1+t)^2}
\end{aligned} \tag{26}
$$

The second line of Equation (26) can be treated as the combination of incomplete beta function with the parameter of incomplete beta function, $t_1 = \frac{H^{-1/\xi}}{1+H^{-1/\xi}}$ and $t_2 = \frac{h^{-1/\xi}}{1+h^{-1/\xi}}$.

$$
\Psi_4 = B\big(t_1, 1-\xi, 1+\xi\big) - B\big(t_2, 1-\xi, 1+\xi\big) \tag{27}
$$

Since we evaluated all integral terms, grouping the terms with S_t and K, then we obtain a closed form of GLO based put option price,

$$
\begin{aligned}
P_t(K) = K \cdot e^{-r(T-t)} \cdot &\left(\frac{1}{1+h^{-1/\xi}} - \frac{1}{1+H^{-1/\xi}}\right) \\
- S_t \cdot e^{-r(T-t)} \cdot &\left(\left(1-\mu+\frac{\sigma}{\xi}\right)\left(\frac{1}{1+h^{-1/\xi}} - \frac{1}{1+H^{-1/\xi}}\right)\right. \\
&\left. + \frac{\sigma B(t_2, 1-\xi, 1+\xi)}{\xi} - \frac{\sigma B(t_1, 1-\xi, 1+\xi)}{\xi}\right)
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\text{where} \qquad H = 1 + \tfrac{\xi}{\sigma}\left(1 - \tfrac{K}{S_t} - \mu\right), &\quad h = 1 + \tfrac{\xi(x-\mu)}{\sigma} \\
t_1 = \tfrac{H^{-1/\xi}}{1+H^{-1/\xi}}, &\quad t_2 = \tfrac{h^{-1/\xi}}{1+h^{-1/\xi}}
\end{aligned}
$$

3 ANALYSIS OF OPTION PRICES

In contrast with the Black-Scholes model, we derive the option pricing model based on the heavy tailed distribution with three parameters. The aim of this new option pricing model is that to reflect extreme events in financial market. With the addition of shape parameter, GLO distribution can be asymmetric and fat tailed distribution, exhibiting larger skewness and kurtosis than normal distribution. In this chapter, we focus on the terms, $N(d_2)$ from the Black-Scholes model, which is the probabilities of the option expiring in-the-money at the maturity. Under the risk neutral probability measure, the probability of the call and put options will be exercised at the maturity can be calculated as follows:

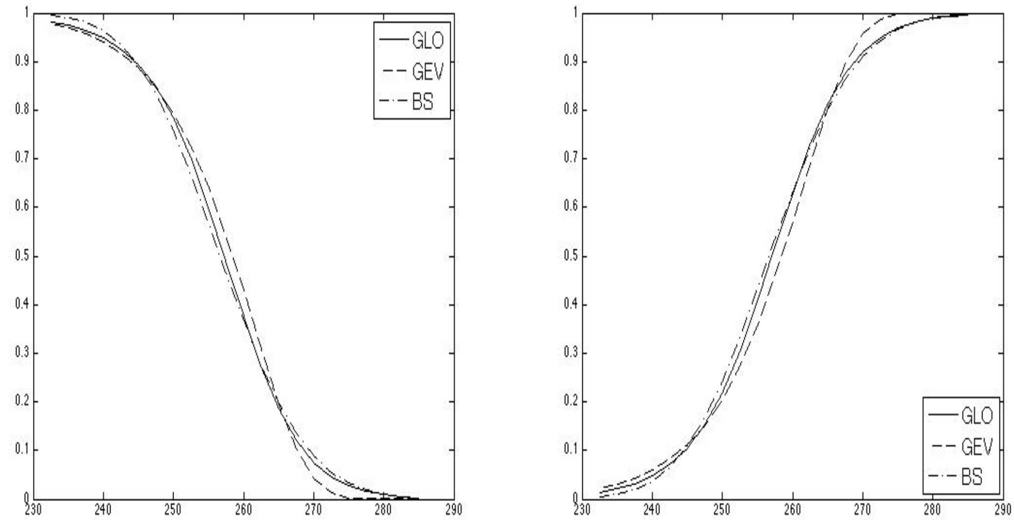
$$P\{S_T > K\} = \int_K^\infty f(S_T) dS_T \tag{29}$$

$$P\{S_T < K\} = \int_0^K f(S_T) dS_T \tag{30}$$

In the GLO call option pricing model, $\left(\frac{1}{1+H^{-1/\xi}}\right)$ corresponds to the $N(d_2)$ term of Black-Scholes model. It is explcitly calculated from Equation (29) and the interpretation of this term is simply that the probability of underlying price will be above the exercise price at maturity. For GLO put option, $\left(\frac{1}{1+h^{-1/\xi}} - \frac{1}{1+H^{-1/\xi}}\right)$ corresponds to $N(-d_2)$ term. The probabilities of option wii be exercised is critical to calculate option price. Remembering that Black-Scholes call option formula is $SN(d_1) - e^{rT}KN(d_2)$, high probability of option will be exercised is related to low exercise price and it results expensive option price. Figure (2a) illustrates in-the-money probabilities of call option for GLO, GEV[1] and Black-Scholes model, respectively. Note that the probability of exercising GLO and GEV call option approaches 1 is much slower than for the Black-Scholes model. Since when shape parameter is positive ($\xi > 0$), implied density

[1]Recently, Markose and Alentorn ([7]) proposed a new option pricing model based on Generalized Extreme Value (GEV) distribution.

function for price is left skewed, resulting in a higher probability of downward moves. Therefore, as the strike price is lowered, it results higher call option prices than those of the Black-Scholes model.



(a) ITM probability for call option

(b) ITM probability for put option

Figure 2: In-The-Money probability from GLO, GEV and BS when $\xi > 0$

However, when higher strike price, the probability of exercising GLO call option moves similar to the Black-Scholes model, while those for GEV call option approaches 0 so fast that it results call prices equal to zero.

Figure (2b) illustrates in-the-money probabilities of put option for each case. The movement of in-the-money probabilities of GLO put option is somewhat similar to the Black-Scholes model. But it is slightly higher when strike price is lower and it concludes that high put option prices. Since the region of low strike price for put option is called the out-of-the money(OTM), OTM put options for GLO and GEV are more expensive than those for the Black-Scholes model, implying that OTM put option prices are much more worth when the left skewed price density function is assumed. It is consistently detected in the option market. OTM put option prices in the market are usually higher than theoretical option prices calculated from the Black-Scholes model.

It is often explained by the skewness of the implied volatility, also 'Volatility Smile'. Hence, with the assumption of the heavy tailed distribution, the inconsistent implied volatilities across the strike price can be also explained.

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