

# Notations in “Testing the Equality of Two Positive-Definite Matrices with Application to Information Matrix Testing”

JIN SEO CHO

School of Economics

Yonsei University

50 Yonsei-ro, Seodaemun-gu, Seoul 120-749, Korea

Email: jinseocho@yonsei.ac.kr

HALBERT WHITE

Department of Economics

University of California, San Diego

9500 Gilman Dr., La Jolla, CA, 92093-0508, U.S.A.

Email: hwhite@weber.ucsd.edu

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## Abstract

We collect the notations defined in “Testing the Equality of Two Positive-Definite Matrices with Application to Information Matrix Testing” by Cho and White (2014).

## Objects to be Estimated

- $\mathbf{A}_* := \mathbf{A}(\boldsymbol{\theta}_*)$ .
- $\mathbf{B}_* := \mathbf{B}(\boldsymbol{\theta}_*)$ .
- $\mathbf{D}_* := \mathbf{B}_* \mathbf{A}_*^{-1}$ .
- $\Sigma_{\mathbf{b},*} := E[\mathbf{b}(Y_t, \mathbf{X}_t) \mathbf{b}(Y_t, \mathbf{X}_t)']$ .
- $T_* := k^{-1} \text{tr}[\mathbf{D}_*] - 1$ .
- $D_* := \det[\mathbf{D}_*]^{1/k} - 1$ .
- $\boldsymbol{\mu}_{\mathbf{a},*} := E[\mathbf{a}(Y_t, \mathbf{X}_t)]$ .
- $\Sigma_{\mathbf{a},*} := E[\mathbf{a}(Y_t, \mathbf{X}_t) \mathbf{a}(Y_t, \mathbf{X}_t)']$ .
- $\boldsymbol{\mu}_{\mathbf{b},*} := E[\mathbf{b}(Y_t, \mathbf{X}_t)]$ .
- $\bar{\mathbf{B}}_* := \bar{\mathbf{B}}(\boldsymbol{\theta}_*)$ .
- $\bar{\mathbf{A}}_* := \bar{\mathbf{A}}(\boldsymbol{\theta}_*)$ .

## Estimators

- $\mathbf{A}_n := \mathbf{A}_n(\boldsymbol{\theta}_n)$ .
- $\mathbf{B}_n := \mathbf{B}_n(\boldsymbol{\theta}_n)$ .
- $\mathbf{D}_n := \mathbf{B}_n \mathbf{A}_n^{-1}$ .
- $\hat{\mathbf{A}}_n := \mathbf{A}_n(\hat{\boldsymbol{\theta}}_n)$ .
- $\hat{\mathbf{B}}_n := \mathbf{B}_n(\hat{\boldsymbol{\theta}}_n)$ .
- $\hat{\mathbf{D}}_n := \hat{\mathbf{B}}_n \hat{\mathbf{A}}_n^{-1}$ .

- $\tilde{\mathbf{A}}_n := \hat{\Sigma}_{\mathbf{a},n} - \hat{\boldsymbol{\mu}}_{\mathbf{a},n} \hat{\boldsymbol{\mu}}_{\mathbf{a},n}'$ .
- $\tilde{\mathbf{B}}_n := \hat{\Sigma}_{\mathbf{b},n} - \hat{\boldsymbol{\mu}}_{\mathbf{b},n} \hat{\boldsymbol{\mu}}_{\mathbf{b},n}'$ .
- $\hat{\boldsymbol{\mu}}_{\mathbf{a},n} := n^{-1} \sum_{t=1}^n \mathbf{a}(Y_t, \mathbf{X}_t)$ .
- $\hat{\Sigma}_{\mathbf{a},n} := n^{-1} \sum_{t=1}^n \mathbf{a}(Y_t, \mathbf{X}_t) \mathbf{a}(Y_t, \mathbf{X}_t)'$ .
- $\hat{\boldsymbol{\mu}}_{\mathbf{b},n} := n^{-1} \sum_{t=1}^n \mathbf{b}(Y_t, \mathbf{X}_t)$ .
- $\hat{\Sigma}_{\mathbf{b},n} := n^{-1} \sum_{t=1}^n \mathbf{b}(Y_t, \mathbf{X}_t) \mathbf{b}(Y_t, \mathbf{X}_t)'$ .
- $\tilde{\mathbf{D}}_n := \tilde{\mathbf{B}}_n \tilde{\mathbf{A}}_n^{-1}$ .
- $\hat{Q}_n := \ln[\text{tr}[\hat{\mathbf{D}}_n]/k]$ .
- $\hat{L}_n := \ln[\det(\hat{\mathbf{D}}_n)]/k$ .

## Test Bases

- $T_n := \text{tr}[\mathbf{D}_n]/k - 1$ .
- $D_n := \det[\mathbf{D}_n]^{1/k} - 1$ .
- $S_n := \text{tr}[\mathbf{D}_n]/k - \det[\mathbf{D}_n]^{1/k}$ .
- $\hat{T}_n := \text{tr}[\hat{\mathbf{D}}_n]/k - 1$ .
- $\hat{D}_n := \det[\hat{\mathbf{D}}_n]^{1/k} - 1$ .
- $\hat{S}_n := \text{tr}[\hat{\mathbf{D}}_n]/k - \det[\hat{\mathbf{D}}_n]^{1/k}$ .
- $\tilde{T}_n := \text{tr}[\tilde{\mathbf{D}}_n]/k - 1$ .
- $\tilde{D}_n := \det[\tilde{\mathbf{D}}_n]^{1/k} - 1$ .
- $\tilde{S}_n := \text{tr}[\tilde{\mathbf{D}}_n]/k - \det[\tilde{\mathbf{D}}_n]^{1/k}$ .
- $\widehat{W}_n := \hat{Q}_n - \hat{L}_n$ .
- $\widehat{M}_n := \hat{T}_n - \hat{L}_n$ .

## Operators

- for  $j = 1, \dots, \ell$ ,  $\partial_j := (\partial/\partial\theta_j)$ .
- for  $i, j = 1, 2, \dots, \ell$ ,  $\partial_{ji}^2 := (\partial^2/\partial\theta_j\partial\theta_i)$ .

## Test Statistics

- $\mathcal{B}_n^{(1)} := nk^2 \left( \frac{1}{2} T_n^2 + \frac{1}{2} D_n^2 \right)$ .
- $\mathcal{B}_n^{(2)} := 2nk \left( \frac{1}{2} T_n^2 + S_n \right)$ .
- $\mathcal{B}_n^{(3)} := 2nk \left( \frac{1}{2} D_n^2 + S_n \right)$ .
- $\widehat{\mathcal{B}}_n^{(1)} := nk^2 \left( \frac{1}{2} \hat{T}_n^2 + \frac{1}{2} \hat{D}_n^2 \right)$ .
- $\widehat{\mathcal{B}}_n^{(2)} := 2nk \left( \frac{1}{2} \hat{T}_n^2 + \hat{S}_n \right)$ .
- $\widehat{\mathcal{B}}_n^{(3)} := 2nk \left( \frac{1}{2} \hat{D}_n^2 + \hat{S}_n \right)$ .
- $\tilde{\mathcal{B}}_n^{(1)} := nk^2 \left( \frac{1}{2} \tilde{T}_n^2 + \frac{1}{2} \tilde{D}_n^2 \right)$ .
- $\tilde{\mathcal{B}}_n^{(2)} := 2nk \left( \frac{1}{2} \tilde{T}_n^2 + \tilde{S}_n \right)$ .
- $\tilde{\mathcal{B}}_n^{(3)} := 2nk \left( \frac{1}{2} \tilde{D}_n^2 + \tilde{S}_n \right)$ .
- $\mathcal{LR}_n^{(1)} := 2\{\ln[L_n(\hat{\boldsymbol{\theta}}_n, \hat{\mathbf{B}}_n)/L_n(\hat{\boldsymbol{\theta}}_n, \hat{\mathbf{A}}_n)]\}$ .
- $\mathcal{LR}_n^{(2)} := 2\{\ln[L_n(\hat{\boldsymbol{\theta}}_n, \hat{\mathbf{B}}_n)/L_n(\hat{\boldsymbol{\theta}}_n, \tilde{d}_n \hat{\mathbf{A}}_n)]\}$ .
- $\mathcal{LR}_n^{(3)} := 2\{\ln[L_n(\tilde{\boldsymbol{\theta}}_n, \tilde{d}_n \hat{\mathbf{A}}_n)/L_n(\tilde{\boldsymbol{\theta}}_n, \hat{\mathbf{A}}_n)]\}$ .

## Supplementary Notations

- $\mathbf{M}_n := \mathbf{A}_*^{-1}(\mathbf{B}_n - \mathbf{A}_n)$ .
- $\mathbf{S}_{j,*} := \mathbf{A}_*^{-1}(\partial_j \mathbf{B}_* - \partial_j \mathbf{A}_*)$ .
- $\mathbf{K}_n := \mathbf{M}_n + \sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}$ .
- $\mathbf{G}_{j,n} := \mathbf{B}_*^{-1} \partial_j (\mathbf{B}_n - \mathbf{B}_*)$ .
- $\mathbf{H}_{j,n} := \mathbf{A}_*^{-1} \partial_j (\mathbf{A}_n - \mathbf{A}_*)$ .
- $\mathbf{J}_{j,n} := \mathbf{G}_{j,n} - \mathbf{H}_{j,n}$ .

- $\mathbf{W}_n := \mathbf{B}_*^{-1}(\mathbf{B}_n - \mathbf{B}_*)$ .
- $\mathbf{U}_n := \mathbf{A}_*^{-1}(\mathbf{A}_n - \mathbf{A}_*)$ .
- $\mathbf{P}_n := \mathbf{W}_n - \mathbf{U}_n$ .
- $\mathbf{R}_{j,*} := \mathbf{B}_*^{-1}\partial_j \mathbf{B}_* - \mathbf{A}_*^{-1}\partial_j \mathbf{A}_*$ .
- $\mathbf{L}_n := \mathbf{P}_n + \sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{R}_{j,*}$ .
- $\mathbf{W}_{o,n} := \mathbf{B}_*^{-1}(\mathbf{B}_n - \mathbf{B}_{*,n})$ .
- $\mathbf{M}_{o,n} := \mathbf{W}_{o,n} - \mathbf{U}_n$ .
- $\mathbf{K}_{o,n} := \mathbf{M}_{o,n} + \sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}$ .
- $\mathbf{G}_{j,o,n} := \mathbf{B}_*^{-1}\partial_j(\mathbf{B}_n - \mathbf{B}_{*,n})$ .
- $\mathbf{J}_{j,o,n} := \mathbf{G}_{j,o,n} - \mathbf{H}_{j,n}$ .
- $\mathbf{N}_* := \mathbf{B}_*^{-1}\tilde{\mathbf{B}}_*$ .
- $\mathbf{C}_{j,*} := \mathbf{B}_*^{-1}\partial_j \tilde{\mathbf{B}}_* - \mathbf{N}_* \mathbf{B}_*^{-1}(\partial_j \mathbf{B}_*)$ .
- $\tilde{\mathbf{M}}_{o,n} := \mathbf{A}_*^{-1}(\tilde{\mathbf{B}}_n - \mathbf{B}_{*,n}) - \mathbf{A}_*^{-1}(\tilde{\mathbf{A}}_n - \mathbf{A}_*)$ .
- $\tilde{\mathbf{M}}_n := \mathbf{A}_*^{-1}(\tilde{\mathbf{B}}_n - \tilde{\mathbf{A}}_n)$ .
- $\tilde{\mathbf{W}}_n := \mathbf{B}_{*,n}^{-1}(\mathbf{B}_n - \mathbf{B}_{*,n})$ .
- $\tilde{\mathbf{P}}_n := \tilde{\mathbf{W}}_n - \mathbf{A}_*^{-1}(\mathbf{A}_n - \mathbf{A}_*)$ .
- $\tilde{\mathbf{R}}_{j,*} := \mathbf{B}_{*,n}^{-1}\partial_j \mathbf{B}_{*,n} - \mathbf{A}_*^{-1}\partial_j \mathbf{A}_*$ .
- $\tilde{\mathbf{L}}_n := \tilde{\mathbf{P}}_n + \sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \tilde{\mathbf{R}}_{j,*}$ .
- $\tilde{\mathbf{G}}_{j,n} := \mathbf{B}_{*,n}^{-1}\partial_j(\mathbf{B}_n - \mathbf{B}_{*,n})$ .
- $\tilde{\mathbf{J}}_{j,n} := \tilde{\mathbf{G}}_{j,n} - \mathbf{H}_{j,n}$ .
- $\tilde{\boldsymbol{\theta}}_n := \hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*$ .
- $\mathbf{Q}_n := (\mathbf{A}_*^{-1}\mathbf{B}_* - \det[\mathbf{D}_*]^{\frac{1}{k}} \mathbf{I})$ .

## Expansions

- $\hat{T}_n^* := \hat{T}_{n,1}^* + \hat{T}_{n,2}^*$ .
- $\hat{T}_{n,1}^* := \frac{1}{k} \text{tr}[\mathbf{K}_n]$ .
- $\hat{T}_{n,2}^* := -\frac{1}{k} \text{tr}[\mathbf{K}_n \mathbf{U}_n] + \frac{1}{k} [\text{tr}[\mathbf{J}_{j,n} - \mathbf{M}_n \mathbf{A}_*^{-1} \partial_j \mathbf{A}_*]]' (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{2k} (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)' \nabla_{\boldsymbol{\theta}}^2 \text{tr}[\mathbf{D}_*] (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$ .
- $\hat{D}_n^* := \hat{D}_{n,1}^* + \hat{D}_{n,2}^*$ .
- $\hat{D}_{n,1}^* := \frac{1}{k} \text{tr}[\mathbf{K}_n]$ .
- $\hat{D}_{n,2}^* := \frac{1}{2k} \left( \frac{1}{k} - 1 \right) \text{tr}[\mathbf{K}_n]^2 + \frac{1}{2k} (\text{tr}[\mathbf{M}_n]^2 + \text{tr}[\mathbf{U}_n^2] - \text{tr}[\mathbf{W}_n^2]) + \frac{1}{k} [\text{tr}[\mathbf{M}_n] \text{tr}[\mathbf{S}_{j,*}]]' (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$   
 $+ \frac{1}{k} [\text{tr}[\mathbf{J}_{j,n} + \mathbf{U}_n \mathbf{A}_*^{-1} \partial_j \mathbf{A}_* - \mathbf{W}_n \mathbf{A}_*^{-1} \partial_j \mathbf{B}_*]]' (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{2k} (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)' \nabla_{\boldsymbol{\theta}}^2 \det[\mathbf{D}_*] (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$ .
- $\hat{S}_n^* := -\frac{1}{2k} \left( \frac{1}{k} - 1 \right) \text{tr}[\mathbf{K}_n]^2 - \frac{1}{2k} (\text{tr}[\mathbf{M}_n]^2 - \text{tr}[\mathbf{M}_n^2]) + \frac{1}{k} \text{tr}[\mathbf{M}_n \sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}]$   
 $- \frac{1}{k} \text{tr}[\mathbf{M}_n] \text{tr}[\sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}] + \frac{1}{2k} \text{tr}[(\sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*})^2] - \frac{1}{2k} \text{tr}[\sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}]^2$ .
- $\hat{T}_{o,n} := \frac{1}{k} \text{tr}[\mathbf{K}_{o,n}(\mathbf{I} - \mathbf{U}_n)] + \frac{1}{k} [\text{tr}[\mathbf{J}_{j,o,n} - \mathbf{M}_{o,n} \mathbf{A}_*^{-1} \partial_j \mathbf{A}_*]]' (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{2k} (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)' \nabla_{\boldsymbol{\theta}}^2 \text{tr}[\mathbf{D}_*] (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$ .
- $\hat{D}_{o,n} := \frac{1}{k} \text{tr}[\mathbf{K}_{o,n}] + \frac{1}{2k} \left( \frac{1}{k} - 1 \right) \text{tr}[\mathbf{K}_{o,n}]^2 + \frac{1}{2k} (\text{tr}[\mathbf{M}_{o,n}]^2 + \text{tr}[\mathbf{U}_n^2] - \text{tr}[\mathbf{W}_{o,n}^2])$   
 $+ \frac{1}{k} [\text{tr}[\mathbf{J}_{j,o,n} + \mathbf{U}_n \mathbf{A}_*^{-1} \partial_j \mathbf{A}_* - \mathbf{W}_{o,n} \mathbf{A}_*^{-1} \partial_j \mathbf{B}_*]]' (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{k} [\text{tr}[\mathbf{M}_{o,n}] \text{tr}[\mathbf{S}_{j,*}]]' (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$   
 $+ \frac{1}{2k} (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)' \nabla_{\boldsymbol{\theta}}^2 \det[\mathbf{D}_*] (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$ .

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$$\begin{aligned}\hat{S}_{o,n} := & -\frac{1}{2k} \left( \frac{1}{k} - 1 \right) \text{tr}[\mathbf{K}_{o,n}]^2 - \frac{1}{2k} (\text{tr}[\mathbf{M}_{o,n}]^2 - \text{tr}[\mathbf{M}_{o,n}^2]) + \frac{1}{k} \text{tr}[\mathbf{M}_{o,n} \sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}] \\ & - \frac{1}{k} \text{tr}[\mathbf{M}_{o,n}] \text{tr}[\sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}] + \frac{1}{2k} \text{tr}[(\sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*})^2] - \frac{1}{2k} \text{tr}[\sum_{j=1}^{\ell} (\hat{\theta}_{j,n} - \theta_{j,*}) \mathbf{S}_{j,*}]^2.\end{aligned}$$

•  $\alpha_n := \alpha_n^{(1)} + \alpha_n^{(2)}.$

•  $\alpha_n^{(1)} := \frac{1}{k} \text{tr}[\mathbf{Q}_n \mathbf{L}_n].$

•  $\alpha_n^{(2)} := \frac{1}{k} [\text{tr}[\mathbf{Q}_n \mathbf{J}_{j,n}]]' \tilde{\boldsymbol{\theta}}_n - \frac{1}{k} \tilde{\boldsymbol{\theta}}_n' [\text{tr}[\mathbf{Q}_n \mathbf{R}_{j,*} \mathbf{A}_*^{-1} \partial_i \mathbf{A}_* + \mathbf{Q}_n (\mathbf{B}_*^{-1} \partial_{ji}^2 \mathbf{B}_* - \mathbf{A}_*^{-1} \partial_{ji}^2 \mathbf{A}_*)]]' \tilde{\boldsymbol{\theta}}_n.$

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$$\begin{aligned}\beta_n := & -\frac{1}{k} \text{tr}[\mathbf{A}_*^{-1} \mathbf{B}_* \mathbf{L}_n \mathbf{U}_n] - \det[\mathbf{D}_*]^{\frac{1}{k}} \left\{ \frac{1}{2k^2} \text{tr}[\mathbf{L}_n]^2 - \frac{1}{2k} \text{tr}[\mathbf{W}_n^2 - \mathbf{U}_n^2] \right\} - \frac{1}{k} [\text{tr}[\mathbf{A}_*^{-1} \mathbf{B}_* \mathbf{P}_n \mathbf{A}_*^{-1} \partial_j \mathbf{A}_*]]' \tilde{\boldsymbol{\theta}}_n \\ & + \frac{1}{k} \det[\mathbf{D}_*]^{\frac{1}{k}} [\text{tr}[\mathbf{U}_n \mathbf{R}_{j,*} + \mathbf{P}_n \mathbf{B}_*^{-1} \partial_j \mathbf{B}_*]]' \tilde{\boldsymbol{\theta}}_n + \frac{1}{2k} \det[\mathbf{D}_*]^{\frac{1}{k}} \tilde{\boldsymbol{\theta}}_n' [\text{tr}[\mathbf{R}_{j,*} \mathbf{R}_{i,*}]] \tilde{\boldsymbol{\theta}}_n.\end{aligned}$$

## References

CHO, J.S. AND WHITE, H. (2014): “Testing the Equality of Two Positive-Definite Matrices with Application to Information Matrix Testing,” Discussion Paper, School of Economics, Yonsei University.