연세대학교 상경대학

경제연구소

Economic Research Institute

Yonsei Universit

서울시 서대문구 연세로 50

50 Yonsei-ro, Seodaemun-gu, Seoul, Korea

TEL: (+82-2) 2123-4065 FAX: (+82-2) 364-9149

E-mail: yeri4065@yonsei.ac.kr

http://yeri.yonsei.ac.kr/new

Testing for Autocorrelation in Quantile Regression Models

Lijuan Huo

Beijing Institute of Technology

Tae-Hwan Kim

Yonsei University

Yunmi Kim

University of Seoul

Dong Jin Lee

University of Connecticut

Decemver 2014

2014RWP-76

Economic Research Institute

Yonsei University

Testing for Autocorrelation in Quantile Regression Models

Lijuan Huo[a], Tae-Hwan Kim[b,*], Yunmi Kim[c] and Dong Jin Lee[d]

[a]School of Humanities and Social Sciences, Beijing Institute of Technology, China
[b]School of Economics, Yonsei University, Korea
[c]Department of Economics, University of Seoul, Korea
[d]Department of Economics, University of Connecticut, USA

September 2014

Abstract: Quantile regression (QR) models have been increasingly employed in many applied areas in economics. At the early stage, applications in the quantile regression literature have usually used cross-sectional data, but the recent development has seen an increase in the use of quantile regression in both time-series and panel datasets. However, testing for possible autocorrelation, especially in the context of time-series models, has received little attention. As a rule of thumb, one might attempt to apply the usual Breusch-Godfrey LM test to the residuals of a baseline quantile regression. In this paper, we demonstrate analytically and by Monte Carlo simulations that such an application of the LM test can result in potentially large size distortions, especially in either low or high quantiles. We then propose a correct test (named the QF test) for autocorrelation in quantile regression models, which does not suffer from size distortion. Monte Carlo simulations demonstrate that the proposed test performs fairly well in finite samples, across either different quantiles or different underlying error distributions.

Keywords: Quantile regression, autocorrelation, LM test.

JEL classifications: C12, C22.

*Corresponding author. Yonsei University, School of Economics, 134 Shinchon-dong, Seodaemun-gu, Seoul 120-749, Korea; Tel.: +82-2-2123-5461; fax: +82-2-2123-8638; e-mail address: tae-hwan.kim@yonsei.ac.kr. This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government(NRF-2014S1A5A2A01010546)

1 Introduction

Since the groundbreaking work of Koenker and Bassett (1978), quantile regression models have been increasingly used in many applied areas in economics due to their flexibility to allow researchers to investigate the relationships between economic variables not only at the center, but also over the entire conditional distribution of the dependent variable. In the early stage of the quantile regression research, the main development in both theory and application occurred mainly in the context of cross-sectional data. However, the application of quantile regression has subsequently moved into the areas of time-series as well as panel data. Some relevant papers are Koul and Mukherjee (1994), Koul and Saleh (1995), Koenker and Xiao (2004), Galvao et al. (2008), Galvao (2009), Xiao (2009), Galvao et al. (2009), Greenwood-Nimmo et al. (2014), and Cho et al. (2014) in the time-series domain, and Koenker (2004), Geraci and Bottai (2007), Abrevaya and Dahl (2008), Lamarche (2010), and Galvao (2011) in the panel data setting.

When quantile regression is applied to time-series models, it is particularly important to check for any sign of autocorrelation because, as correctly pointed out by Koenker (2005, pp. 128), the typical IID (independent and identically distributed errors) condition is usually imposed on the regression error terms in both mean-regression and quantile regression. If the error terms are serially correlated in quantile models, the effect is the same as in any mean model; that is, employing the usual variance-covariance matrix for inference becomes invalid, as shown in Weiss (1990) in the case of least absolute deviations (LAD) regression.

In the quantile regression literature, there has been some effort to investigate the asymptotic properties of the LAD estimator and to derive the correct asymptotic variance-covariance matrix of the LAD estimator in the presence of autocorrelation. For example, see Davis and Dunsmuir (1997) and Weiss (1990, 1991). However, little attention has been paid to the issue of testing for autocorrelation in quantile regression models. The only significant exceptions are the works of Weiss (1990) and Furno (2000). Weiss (1990) investigates how to test for AR(1) serial correlation in LAD regression models, while Furno (2000) studies a testing procedure for random coefficient autocorrelated (RCA) errors based on LAD residuals.

In the absence of any theoretical prescription that can be used in quantile regression, one might attempt, as a rule of thumb, to apply the usual LM test to the residuals from the baseline quantile regression. In this paper, we demonstrate that such an application of the LM test can result in potentially large size distortions, especially in either low or high quantiles. For example, when the error distribution is the standard normal, the empirical rejection rates for the null hypothesis of no autocorrelation are around 30% for both 0.05 and 0.95 quantiles when the nominal size is 5%. Given that quantile models are usually regarded as a supplementary tool for OLS-type mean regression and therefore are mainly used to investigate either low and high quantiles, such severe size distortions can be a serious problem. After documenting the phenomenon of spurious autocorrelation induced by the usual LM test, we propose a correctly-sized test (named the QF test) for autocorrelation in quantile regression models, which does not suffer from size distortion. Monte Carlo simulations demonstrate that the proposed test performs fairly well in finite samples, across either different quantiles or different underlying error distributions.

This paper is organized as follows. In section 2, we demonstrate the effect of blindly applying the LM test to quantile regression models and provide a theoretical explanation for such a distortion. Then, a correctly-sized test for autocorrelation in quantile regression models is proposed in Section 3, and its asymptotic distribution is derived. In Section 4, we carry out Monte Carlo simulations to investigate the finite sample properties of the proposed test. Section 5 provides an empirical example as an illustration for the proposed test. Concluding remarks are in Section 6.

2 Spurious Autocorrelation in Quantile Regression Models

We start with the linear model with AR(p) errors considered by Weiss (1990) for LAD estimation:

$$\begin{aligned} y_t &= x_t'\beta_\theta + \epsilon_{\theta t}, \\ \epsilon_{\theta t} &= \rho_1\epsilon_{\theta t-1} + \rho_2\epsilon_{\theta t-1} + ... + \rho_p\epsilon_{\theta t-p} + v_{\theta t}, \end{aligned} \tag{1}$$

where x_t is a $k \times 1$ vector of explanatory variables whose first column is one, T denotes the number of observations, $\theta \in (0, 1)$ is a pre-specified quantile level, and β_θ is the vector of unknown quantile parameter to be estimated. Weiss (1990) studies the LAD case with $\theta = 1/2$ and $p = 1$. Note that the vector (x_t) of explanatory variables can include some lags of the dependent variable.[1] We assume that $v_{\theta t}$'s are independently distributed. More regularity conditions on the error terms will be specified in detail later. We also assume that there is no endogenous variable in x_t.

The primary null hypothesis that we wish to test for throughout the paper is given by

$$H_0 : \rho_1 = \rho_2 = ...\rho_p = 0. \tag{2}$$

Since autocorrelation is a common phenomenon for time series data, it might be possible that some ρ_i's are not zero. In the presence of autocorrelated errors, quantile estimates become less efficient, and more importantly any inference based on the conventional variance-covariance matrix becomes invalid, as shown in Weiss (1990) for the case of LAD regression. Therefore, the importance of checking for autocorrelation in quantile models is the same as in OLS-type mean regression models. Nevertheless, no formal investigation into this important issue has been carried out so far. In the absence of any theoretical prescription that can be used in quantile regression, one might attempt, as a rule of thumb, to apply the usual LM test to the residuals of a baseline quantile regression. We will show that this rule-of-thumb approach can result in potentially significant size distortions (i.e., spurious autocorrelation).

We first demonstrate the phenomenon of spurious autocorrelation caused by the naive application of the LM test. The standard LM test is based on the quantile residuals ($e_{\theta t}$) obtained from the following quantile regression:

$$y_t = x_t'\hat{\beta}_\theta + e_{\theta t},$$

where $\hat{\beta}_\theta$ is the quantile regression estimator. Once the quantile residuals $e_{\theta t}$ are obtained, the auxiliary regression of $e_{\theta t}$ on x_t and $e_{\theta t-1}, e_{\theta t-2}, \cdots, e_{\theta t-p}$ is carried out as follows:

$$e_{\theta t} = x_t'\hat{\gamma}_0 + \hat{\rho}_1 e_{\theta,t-1} + \hat{\rho}_2 e_{\theta,t-2} + \cdots + \hat{\rho}_p e_{\theta,t-p} + \hat{v}_t, \tag{3}$$

where $\hat{v}_t$'s are the residuals from the auxiliary regression. The LM statistic denoted as LM_T is given by $T \times R^2$, where R^2 is the R-square from the auxiliary regression in (3), and the LM statistic is compared with an appropriate critical value from the chi-squared distribution with p degrees of freedom (DF).

Suppose that the error terms in (1) are not serially correlated; i.e. $E(\epsilon_{\theta t}\epsilon_{\theta s}) = 0$ for $t \neq s$. Then, any rejection of the null of no autocorrelation larger than a pre-specified significance level indicates the phenomenon of spurious regression. To illustrate the performance of the LM test explained above, we specify our data generation process (DGP) and simulation setup as follows. We assume a linear model and include only one scalar exogenous variable (z_t) in the model so that the DGP is given by:

$$y_t = \beta_{0\theta} + \beta_{1\theta}w_t + \epsilon_{\theta t}, \tag{4}$$

[1] We will assume throughout the paper that the obvious stationarity condition is imposed for both y_t and $\epsilon_{\theta t}$.

where w_t are independently generated from the standard normal distribution. To focus on the size property of the LM test, the error terms $\epsilon_{\theta t}$ are also independently generated from a distribution. We consider three different distributions to generate the error terms, which are the standard normal distribution $N(0,1)$, and student t-distribution with five degrees of freedom $t(5)$ to represent thin and moderately heavy tailed symmetric distributions, respectively, and lastly the lognormal distribution with $\mu = 1, \sigma = 0.4$ (centered at zero by subtracting the mean value) denoted by $LN(1, 0.4)$ to represent an asymmetric distribution.

Once w_t and $\epsilon_{\theta t}$ are generated as described above, the dependent variable y_t is generated through (4) with $\beta_{0\theta} = 1$ and $\beta_{1\theta} = 1$ and with various sample sizes $T = 50, 300, 500, 1000, 3000, 5000$. For our initial simulations, we choose $p = 2$ for the auxiliary regression in (3) although we study the effect of varying the value of p later. The rejection probability (i.e., the empirical size) of the LM test is calculated through 1,000 replications, and the nominal size is fixed at 5% in all simulations. Table 1 shows the empirical sizes of the LM test for various quantile indexes $\theta = 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95$. We also include the empirical sizes of the LM test based on LS regression for the purpose of comparison. Table 1 consists of three panels, each of which corresponds to one of the three error distributions ($N(0,1), t(5)$, and $LN(1, 0.4)$). In each panel, sample sizes are in rows, while each column represents a different quantile index (θ).

We first consider the $N(0,1)$ case displayed in the top panel in Table 1. When the LM test is carried out for LS regression, the empirical sizes are fairly close to the nominal size 5%, regardless of the sample size, which is expected and well-known. However, when the LM test is implemented for quantile models, we observe very different results. To begin with, when based on median (or LAD) regression with $\theta = 0.5$, the rejection frequencies, although slightly over-sized, are relatively close to the nominal size. However, as the quantile index θ moves toward either low or high quantiles, the size distortion increases symmetrically around $\theta = 0.5$. For example, the rejection probability is about 20% for $\theta = 0.1, 0.9$, but it increases even further to around 30% for $\theta = 0.05, 0.95$. It is interesting to note that size distortions do not seem to decrease or to vanish as the sample size increases, even up to $T = 5,000$. The second panel of Table 1 shows the rejection frequencies of the LM test when the error term is generated by the moderately fat-tailed distribution $t(5)$. The results are quite similar to the $N(0,1)$ case, except that size distortions become more pronounced at extreme quantiles like $\theta = 0.05, 0.95$ where the empirical sizes are in the range of 35% to 45%.

The bottom panel of Table 1 shows the results when the error term is generated from the asymmetric lognormal distribution $LN(1, 0.4)$. The results are in sharp contrast to all of the previous cases in that the empirical sizes are no longer symmetric around the median $\theta = 0.5$. The rejection probability increases moderately as θ moves to lower quantiles (e.g., it is about 14% when $\theta = 0.05$; an increase of 6%p from the median value), whereas it increases sharply as θ moves toward the high end of the distribution (e.g., the rejection probability is about 50% when $\theta = 0.95$; an increase of 42%p from the median value).

Thus far, we have fixed the number of lags used in the auxiliary regression at two ($p = 2$) in all simulations. One might be curious on the effect of using different values for p. We take the results with $\theta = 0.9$ from Table 1 as the baseline case and attempt to use other values of p; specifically $p = 1, 2, 4$, and 12, which are typically used in applications depending on the frequency (monthly, quarterly, or yearly) of data. The results are reported in Table 2.

It is well known that using different values of p does not have any effect on the size property of the LM test when it is based on LS regression. This is confirmed in the first four columns in Table 2. The rejection probability does not vary with p for LS regression. The only exception occurs when $p = 12$ and $T = 50$. For example, the empirical size is 1.5% for the $N(0,1)$ case with $p = 12$ and $T = 50$. This must be due to the small number of degrees of freedom in the auxiliary regression for the LM test because there are too many parameters to be estimated relative to the sample size.

The invariance property of the LM test with respect to the value of p, however, breaks down for quantile regression as clearly shown in columns 5-8 in Table 2. A common pattern over all the three error distributions is that the rejection probability tends to decrease as p increases. For example, the rejection probability with $T = 500$ in the case of the $t(5)$ error distribution is given by 28.6%, 21,8%, 17.5% and 10.7% for $p = 1, 2, 4$, and 12, respectively. Therefore, the size distortion problem in the LM test used in quantile regression models can be lessened by employing a large value of p. However, not only will it not completely eliminate the problem, but it is also likely to cause a reduction in power of the LM test.

We now theoretically explain why such a phenomenon of spurious autocorrelation occurs. The main reason is that the asymptotic distribution of the LM statistic LM_T is not the usual chi-squared distribution with p degrees of freedom. In deriving the correct asymptotic distribution of LM_T, we will distinguish two cases: (i) the quantile error $\epsilon_{\theta t}$ is heteroscedastic in the sense that $E[\epsilon_{\theta t}|x_t] = x_t'\gamma_0$ for some value of γ_0, and (ii) $\epsilon_{\theta t}$ is homoscedastic in the sense that $E[\epsilon_{\theta t}|x_t]$ is constant. Obviously, the latter is a special case of the former. To formerly derive the correct asymptotic distribution of LM_T, we first impose the following standard assumptions. In all assumptions, lemmas, theorems and calculations below, we will impose the primary null hypothesis in (2) even when it is not explicitly stated. Thus, the assumptions imposed on $\epsilon_{\theta t}$ below are the same as for $\nu_{\theta t}$ under the null hypothesis.

Assumption 1 (i) Given $\theta \in (0, 1)$, the true quantile parameter β_θ is an interior point in the parameter space Θ.
(ii) The quantile error $\epsilon_{\theta t}$ is independent of x_{t-i} for $i > 0$ and satisfies the quantile version of the orthogonality condition $E(\psi_\theta(\epsilon_{\theta t})|x_t) = 0$ almost surely for every t where $\psi_\theta(u) = \theta - 1_{[u\leq 0]}$.
(iii) The distribution function of $\epsilon_{\theta t}$ conditional on x_t has a continuous density $f_{\epsilon_{\theta t}}(\cdot)$ that satisfies $0 < f_{\epsilon_{\theta t}}(0) < \infty$ for all $t = 1, \ldots, T$.
(iv) The sequence $\{(x_t, \epsilon_{\theta t})\}$ is mixing with either ϕ of size $-r/2(r-1)$, $r \geq 2$ or α of size $-r/(r-2)$, $r > 2$.
(v) Let $z_t = (x_t', \tilde{\epsilon}_{\theta t}')'$ where $\tilde{\epsilon}_{\theta t} = (\epsilon_{\theta t-1}, \ldots, \epsilon_{\theta t-p})'$. Then, $E|z_{ti}|^{2(r+\delta)} < \Delta < \infty$ for some $\delta > 0$, and for all $i = 1, \ldots, k + p$ and t.
(vi) $E[|z_{ti}\epsilon_{\theta t}|^r] < \infty$ for every t.
(vii) $E[\frac{1}{T}\sum_{t=1}^{T} z_t z_t']$ is positive definite uniformly in T.

When we derive the asymptotic distribution of the LM statistic, we will need to deal with the probability limit of two sample moments M_{Tz} and Q_T, which are respectively defined as:

$$\begin{aligned} M_{Tz} &= \frac{1}{T}\sum z_t z_t' = \begin{pmatrix} \frac{1}{T}\sum x_t x_t' & \frac{1}{T}\sum x_t \tilde{\epsilon}_{\theta t}' \\ \frac{1}{T}\sum \tilde{\epsilon}_{\theta t} x_t' & \frac{1}{T}\sum \tilde{\epsilon}_{\theta t}\tilde{\epsilon}_{\theta t}' \end{pmatrix}, \\ Q_T &= \begin{pmatrix} \frac{1}{T}\sum \psi_\theta(\epsilon_{\theta t})^2 x_t x_t' & \frac{1}{T}\sum \epsilon_{\theta t}^* \psi_\theta(\epsilon_{\theta t}) x_t z_t' \\ \frac{1}{T}\sum \epsilon_{\theta t}^* \psi_\theta(\epsilon_{\theta t}) z_t x_t' & \frac{1}{T}\sum \epsilon_{\theta t}^{*2} z_t z_t' \end{pmatrix}, \end{aligned}$$

where $\epsilon_{\theta t}^* = \epsilon_{\theta t} - E[\epsilon_{\theta t}|x_t]$. Under Assumption 1, we can apply the law of large numbers for mixing sequences to obtain the following:

$$\begin{aligned} M_{Tz} \xrightarrow{p} M_z &= \begin{pmatrix} M_x & M_{x\epsilon} \\ M_{x\epsilon}' & M_\epsilon \end{pmatrix}, \\ Q_T \xrightarrow{p} Q &= \begin{pmatrix} \theta(1-\theta)M_x & M_{zx,\epsilon} \\ M_{zx,\epsilon}' & M_{z,\epsilon} \end{pmatrix}, \end{aligned}$$

where $M_x = \lim_{T\to\infty} E[\frac{1}{T}\sum x_t x_t']$, $M_{x\epsilon} = \lim_{T\to\infty} E[\frac{1}{T}\sum x_t \tilde{\epsilon}_{\theta t}']$, $M_\epsilon = \lim_{T\to\infty} E[\frac{1}{T}\sum \tilde{\epsilon}_{\theta t}\tilde{\epsilon}_{\theta t}']$, $M_{zx,\epsilon} = \lim_{T\to\infty} E[\frac{1}{T}\sum \epsilon_{\theta t}^* \psi_\theta(\epsilon_{\theta t}) x_t z_t']$, and $M_{z,\epsilon} = \lim_{T\to\infty} E[\frac{1}{T}\sum \epsilon_{\theta t}^{*2} z_t z_t']$. We note that M_ϵ is not necessarily a diagonal matrix as in the LS framework because $E(\tilde{\epsilon}_{\theta t})$ is not necessarily zero. We also note that $M_{zx,\epsilon}$ can be decomposed as follows:

$$M_{zx,\epsilon} = \begin{pmatrix} M_{x,\epsilon} & M_{x\epsilon,\epsilon} \end{pmatrix},$$

where $M_{x,\epsilon} = \lim_{T\to\infty} E[\frac{1}{T}\sum \epsilon_{\theta t}^* \psi_\theta(\epsilon_{\theta t}) x_t x_t']$ and $M_{x\epsilon,\epsilon} = \lim_{T\to\infty} E[\frac{1}{T}\sum \epsilon_{\theta t}^* \psi_\theta(\epsilon_{\theta t}) x_t \tilde{\epsilon}_{\theta t}']$.

Based on the previous two results, we provide the following lemma which is useful in proving the main theorem on the asymptotic distribution of the LM statistic LM_T. In the lemma, $0_{m,n}$ is the $m \times n$ matrix of zeros and I_n is the $n \times n$ identity matrix.

Lemma 1. Suppose that (i) Assumption 1 holds and (ii) $\epsilon_{\theta t}$ is heteroscedastic. Let $\hat{\gamma} = (\hat{\gamma}_0', \hat{\rho}_1, \ldots, \hat{\rho}_p)'$. Define $\gamma = (\gamma_0' \;\; 0_p')'$ where γ_0 is the $k \times 1$ vector such that $E[\epsilon_{\theta t}|x_t] = x_t'\gamma_0$. Then, the asymptotic distribution of $\hat{\gamma}$ under the null hypothesis in (2) is given by:

$$\sqrt{T}(\hat{\gamma} - \gamma) \xrightarrow{d} N(0, A + M_z^{-1} M_{z,\epsilon} M_z^{-1}),$$

where

$$\begin{aligned}
A &= \begin{pmatrix} A_1 + A_2 & A_3 \\ A_3' & 0_{p,p} \end{pmatrix}, \\
A_1 &= \theta(1-\theta)(P_0 M_x - M_x^{-1} M_{x\epsilon} P M_{x\epsilon}') D^{-1} M_x D^{-1} (P_0 M_x - M_x^{-1} M_{x\epsilon} P M_{x\epsilon}')', \\
A_2 &= (P_0 M_x - M_x^{-1} M_{x\epsilon} P M_{x\epsilon}') D^{-1} (P_0 M_{x,\epsilon} - M_x^{-1} M_{x\epsilon,\epsilon} P M_{x\epsilon,\epsilon}')', \\
A_3 &= -(P_0 M_x - M_x^{-1} M_{x\epsilon} P M_{x\epsilon}') D^{-1} (-P M_{x\epsilon}' M_x^{-1} M_{x,\epsilon} + P M_{x\epsilon,\epsilon}')', \\
P_0 &= (M_x - M_{x\epsilon} M_\epsilon^{-1} M_{x\epsilon}')^{-1}, \\
P &= (M_\epsilon - M_{x\epsilon}' M_x^{-1} M_{x\epsilon})^{-1}, \\
D &= \lim_{T\to\infty} \frac{1}{T} \sum E[f_{\epsilon_{\theta t}}(0)^2 x_t x_t'].
\end{aligned}$$

All technical proofs are provided in the Appendix. Using Lemma 1, we can prove that the LM statistic either diverges to infinity or weakly converges to a distribution that is different from the chi-squared distribution with p degrees of freedom, depending on the heteroscedastic nature of the quantile error $\epsilon_{\theta t}$.

Theorem 1. Suppose that Assumption 1 holds.
(i) If $\epsilon_{\theta t}$ is heteroscedastic, then

$$LM_T \xrightarrow{p} \infty.$$

(ii) If $\epsilon_{\theta t}$ is homoscedastic, then

$$LM_T \xrightarrow{d} B'(R_1 M_z^{-1} R_1')^{-1} B,$$

where $B \sim N\left(0_{k+p-1,1}, R_1(A + M_z^{-1} M_{z,\epsilon} M_z^{-1}) R_1'\right)$ and $R_1 = (0_{k+p-1,1}, I_{k+p-1})$.

3 A Correctly-Sized Test

In this section, we propose an alternative test for autocorrelation that can be used in quantile regression models without size distortion. The proposed test is the usual F test for the null hypothesis that all of the parameters for the lagged residuals in the auxiliary regression in (3) are jointly zero. The proposed test is called the QF test.

The test statistic (denoted by QF_T) for the QF test for this null hypothesis in (2) is given by

$$QF_T = \frac{(\sum_{t=1}^T \tilde{v}_t^2 - \sum_{t=1}^T \hat{v}_t^2)}{\sum_{t=1}^T \hat{v}_t^2/(T-p-k)}, \tag{5}$$

where $\hat{v}_t$ is the residuals from the unrestricted auxiliary regression in (3), and $\tilde{v}_t$ is the residuals from the restricted auxiliary regression where the null hypothesis is imposed. The following theorem demonstrates that the QF statistic is asymptotically distributed as the chi-squared distribution with p degrees of freedom.

Theorem 2. Suppose that (i) Assumption 1 holds and (ii) $\epsilon_{\theta t}$ is homoscedastic. Then, we have that

$$QF_T \xrightarrow{d} \mathcal{X}_p^2,$$

provided that the null hypothesis (2) is correct.

Using elementary matrix manipulations, the QF statistic in (5) can be equivalently written as the Wald representation:

$$QF_T = (\sqrt{T}R_2\hat{\gamma})(R_2\tilde{M}_{Tz}^{-1}R_2')^{-1}(\sqrt{T}R_2\hat{\gamma})/s^2, \tag{6}$$

where $R_2 = (0_{p,k}, I_p)$, $\tilde{M}_{Tz} = \frac{1}{T}\sum_{t=1}^T \tilde{z}_t\tilde{z}_t'$, $\tilde{z}_t = (x_t', e_{\theta t-1}, \ldots, e_{\theta t-p})'$ and $s^2 = \sum_{t=1}^T \hat{v}_t^2/(T-p-k)$. Theorem 2 is valid only under homoscedasticity. If the errors are heteroscedastic, then the correct Wald statistic (denoted as QF_T^H) is obtained using the typical robust variance-covariance estimator as follows:

$$QF_T^H = (\sqrt{T}R_2\hat{\gamma})(R_2\hat{V}R_2')^{-1}(\sqrt{T}R_2\hat{\gamma})', \tag{7}$$

where $\hat{V} = \tilde{M}_{Tz}^{-1}M_{Tz,\epsilon}\tilde{M}_{Tz}^{-1}$, and $M_{Tz,\epsilon} = \frac{1}{T}\sum_{t=1}^T e_{\theta t}^2\tilde{z}_t\tilde{z}_t'$. The following theorem shows that QF_T^H converges to $\mathcal{X}_p^2$ in distribution under the null hypothesis under consideration.

Theorem 3. Suppose that (i) Assumption 1 holds and (ii) $\epsilon_{\theta t}$ is heteroscedastic. Then, we have that

$$QF_T^H \xrightarrow{d} \mathcal{X}_p^2\ ,$$

provided that the null hypothesis (2) is correct.

We note that, although Theorem 3 is stated and proved under heteroscedasticity, it is still valid even under homoscedasticity because the difference between the two variance-covariance terms in (6) and (7) (i.e., $R_2M_{Tz}^{-1}R_2'$ and $R_2M_{Tz}^{-1}M_{Tz,\epsilon}M_{Tz}^{-1}R_2'$) becomes negligible as the sample size increases.

4 Monte Carlo Simulations

In this section, we investigate the finite sample properties of the proposed test. We also compare the proposed test with an extension of the test for autocorrelation proposed in Weiss (1990). Although Weiss (1990) focuses on both the LAD regression with $\theta = 0.5$ and AR(1) errors, we conjecture that his test can be straightforwardly extended to quantile regression models by simply replacing the sign function in his argument with our step function $\psi_\theta(e_{\theta t})$. Based on Weiss (1990), we consider the following auxiliary regression:

$$\psi_\theta(e_{\theta t}) = x_t'\tilde{\gamma}_0 + \tilde{\rho}_1 e_{\theta,t-1} + \tilde{\rho}_2 e_{\theta,t-2} + \cdots + \tilde{\rho}_p e_{\theta,t-p} + \tilde{\eta}_t. \tag{8}$$

The modified auxiliary regression in (8) is the same as the one in (3) except that the dependent variable is not the residual itself $e_{\theta t}$, but the transformed residual $\psi_\theta(e_{\theta t})$. The test given by $T \times R^2$, where R^2 is the R-square from the modified auxiliary regression in (8), is called the QR-LM test. Under the null hypothesis that $\rho_1 = \rho_2 = ...\rho_p = 0$, Weiss (1990) shows that the QR-LM statistic with $\theta = 0.5$ follows asymptotically the chi-squared distribution with p degrees of freedom. Our simulations indicates that the asymptotic distribution is the same for other values of $\theta \in (0, 1)$.

For comparability, we employ the identical simulation setup as in the previous section. The simulation results for the proposed QF test in quantile models are shown in Table 3 along with its counterpart in LS regression. It is clearly demonstrated that the QF test does not show any sign of size distortion for all three error distributions. As before, the effect of changing p is also simulated, and the results shown in Table 4 confirm the desired invariance property of the QF test with respect to the number of lagged residuals in the auxiliary regression. The performance of the QR-LM test is reported in Table 5. As shown in Table 5, the QR-LM test performs quite well in terms of size, regardless of the choice of quantile index or error distribution. The desired invariance property with respect to p for the QR-LM test is demonstrated in Table 6. It can be concluded that the two tests are comparable in terms of size.

We now compare the two tests in terms of power. The simulation setup is identical to the previous cases, except that the error $\epsilon_{\theta t}$ follows an AR(1) process:

$$\epsilon_{\theta t} = \rho \epsilon_{\theta t-1} + \eta_t \tag{9}$$

where η_t is drawn from previous three distributions ($N(0, 1)$, $t(5)$, and $LN(1, 0.4)$).

Tables 7 and 8 show the rejection probabilities of the QF test ($p = 2$) with $\rho = 0.4$ and $\rho = 0.8$, respectively. Table 7 shows that rejection frequencies for three distributions are quite similar and all rejection frequencies quickly increase to 100% as the sample size increases from $T = 50$ to $T = 300$. The more correlated case with $\rho = 0.8$ is reported in Table 8, and, as expected, fairly high rejection frequencies are shown even for the smallest sample size of $T = 50$.

The analogous results for the QR-LM test are displayed in Table 9 with $\rho = 0.4$ and Table 10 with $\rho = 0.8$. It is clearly seen in Table 9 that the QR-LM test is inferior to the QF test in terms of power. The rejection frequencies of the QR-LM test are incommensurably smaller than those of the QF test. For example, the rejection probability of the QR-LM test for the $t(5)$ case with $\theta = 0.05$ and $T = 100$ is 29.1%, while the QF test delivers 91.7% for the same case. It is interesting to note that the QR-LM test tends to be more powerful at the middle quantiles than at either lower or upper quantiles. For example, the rejection probability of the QR-LM test for the $t(5)$ case with $T = 100$ increases from 29.1% to 87.2% when θ changes from 0.05 to 0.5. However, the rejection probabilities of the QR-LM test even at the middle quantiles are much smaller than the corresponding rejection probabilities of the QF test.

5 An Empirical Example

Fama and French (1996) propose the following three-factor model:

$$r_t = \beta_0 + \beta_1 r_t^m + \beta_2 r_t^{SMB} + \beta_3 r_t^{HML} + \epsilon_t, \tag{10}$$

where r_t is the excess return of an asset, r_t^m is the excess return on the market portfolio, r_t^{SMB} is the difference between two portfolios of small stocks and of large stocks (SMB, small minus big), and r_t^{HML} is the difference between two portfolios of low book-to-market stocks and of high book-to-market stocks (HML, high minus low). In computing excess returns, the three-month treasury bill is used as an approximation of the risk-free rate.

Allen et al. (2009) estimate the three-factor model in (10) using quantile regression to reveal how the impacts of these factors on the excess return of an asset can vary across quantiles. We use the same model as an illustration for the proposed test. We set up a data set of the monthly return of Microsoft Corporation (MSFT) in Nasdaq, three factors collected from Professor Ken French's website, and three-month treasury bill as the risk free rate downloaded Federal Reserve Bank of st.Louis site. The sample period is from July 2006 to June 2012. Given that stock returns are not usually serially correlated, we use $p = 1$ in the auxiliary regression to compute the conventional LM test as well as the proposed QF test. Table 11 shows a range of quantile estimates for the parameters in (10) for $\theta = 0.05$, 0.1, 0.25, 0.5, 0.75, 0.9 and 0.95. For the purpose of comparison, the OLS results are also included in the first column of the table. Table 11 also provides the results of the two tests (the LM test and the QF test) for autocorrelation in quantile regression, including the computed test statistics and the corresponding $p-$values for the selected quantile indexes.

The top panel in Table 11 reports OLS and quantile regression estimation results showing that the impact of the three factors on the return of MSFT seems to vary across quantiles. Such variation is especially noticeable for the SMB factor; the impact of SMB is negative at low-to-midium quantiles but becomes positive at upper quantiles.

Turning to the issue of testing for autocorrelation, the results are reported in the bottom panel of Table 11, which shows the test results from (i) the blind application of the conventional LM test and (ii) the proposed QF test. We first discuss the results from the LM test. Based on OLS regression where the application of the LM test is correct, the null of no autocorrelation is not rejected; the p-value is 0.29. When the LM test is applied to quantile regression, the null hypothesis is rejected for all the selected quantile indexes except for $\theta = 0.75$ and 0.90 at the 5% significant level. Hence, researchers can conclude that there exists strong empirical evidence to support serial correlation in the quantile regression model under consideration. On the other hand, the proposed method rejects the null hypothesis not only in OLS regression, but also at all the chosen quantile indexes; p-values are around 0.25 to 0.55 at all quantiles, which indicates that the test results from the LM test are most likely to be spurious. These results are more or less in line with what has been discovered in the previous sections.

6 Conclusion

Although quantile regression has been increasingly employed in the context of time-series data, no formal investigation of testing for autocorrelation in quantile regression models has been proposed in the literature. In this paper, we first demonstrate the phenomenon of spurious autocorrelation that occurs when the conventional LM test is blindly applied in quantile regression models. We find that the size distortion problem is more pronounced in either low or high quantiles than middle quantiles. We then propose an alternative testing procedure (the QF test) that are free of size distortion. We derive the asymptotic distribution of the proposed test statistic, and Monte Carlo simulations show that the proposed test works well and has good size and power properties even in finite samples.

References

[1] Abrevaya, J., Dahl, C.M., 2008. The effects of birth inputs on birthweight: evidence from quantile estimation on panel data. Journal of Business and Economic Statistics 26, 379-397.

[2] Allen, D.E., Singh, A.K., Powell R., 2009. Asset pricing, the Fama-French factor model and the implications of quantile regression analysis, working paper.

[3] Cho, J.S, Kim, T.-H., Shin, Y., 2014, Quantile cointegration using autoregressive distributed-lag modeling approach, discussion paper.

[4] Davidson, J., 1994. Stochastic limit theory, Oxford.

[5] Davis, R.A., Dunsmuir, W.T.M., 1997. Least absolute deviation estimation for regression with ARMA errors. Journal of Theoretical Probability 10, 481-497.

[6] Fama, E.F., French K.R., 1996. Multifactor explanations of asset pricing anomalies. Journal of Finance 1,55-83.

[7] Furno, M., 2000. LM tests in the presence of non-normal error distributions, Econometric Theory 16, 249-261.

[8] Galvao, A.F., 2009. Unit root quantile autoregression testing using covariates. Journal of Econometrics 152, 165-178.

[9] Galvao, A.F., 2011. Quantile regression for dynamic panel data with fixed effects. Journal of Econometrics 164, 142-157.

[10] Galvao, A.F., Montes-Rojas, G., Olmo, J., 2008. Threshold quantile autoregressive models, discussion paper.

[11] Galvao, A.F., Montes-Rojas, G., Park, S.Y., 2009. Quantile autoregressive distributed lag model with an application to house price returns, discussion paper.

[12] Geraci, M., Bottai, M., 2007. Quantile regression for longitudinal data using the asymmetric Laplace distribution. Biostatistics 8, 140-154.

[13] Godfrey, L.G., 1978. Testing for higher order serial correlation in regression equations when the regressors include lagged dependent variables. Econometrica 47, 1303-1310.

[14] Greenwood-Nimmo, M., Kim, T-H., Shin, Y., Treeck, T., 2014. Fundamental asymmetries in US monetary policymaking: evidence from a nonlinear autoregressive distributed lag quantile regression model, discussion paper.

[15] Koenker, R., 2004. Quantile regression for longitudinal data. Journal of Multivariate Analysis 91, 74-89.

[16] Koenker, R., 2005. Quantile regression. Cambridge University Press.

[17] Koenker, R., Bassett,G., 1978. Regression quantiles, Econometrica 46, 33-49

[18] Koenker, R., Xiao, Z., 2004. Unit root quantile autoregression inference. Journal of the American Statistical Association 99, 775-787.

[19] Komunjer, I., 2005. Quasi-maximum likelihood estimation for conditional quantiles. Journal of Econometrics 128, 137-164.

[20] Koul, H., Mukherjee, K., 1994. Regression quantiles and related processes under long range dependent errors. Journal of Multivariate Analysis 51, 318-337.

[21] Koul, H., Saleh, A., 1995. Autoregression quantiles and related rank score processes. Annals of Statistics 23, 670-689.

[22] Lamarche, C., 2010. Robust penalized quantile regression estimation and inference for panel data. Journal of Econometrics 157, 396-408.

[23] Newey, W.K., McFadden, D.L., 1994. Large sample estimation and hypothesis testing, in R. F. Engle and D. L. Newey (eds.), Handbook of Econometrics 4, 2113-2247, Elsevier Science.

[24] Weiss, A.A., 1990. Least absolute error estimation in the presence of serial correlation. Journal of Econometrics 44, 127-158.

[25] Weiss, A.A., 1991. Estimating nonlinear dynamic models using least absolute error estimation. Econometric Theory 7, 46-68.

[26] White, H., 2000. Asymptotic theory for econometricians, Academic Press.

[27] Xiao, Z., 2009. Quantile cointegrating regression, Journal of Econometrics 150, 248–260.

Appendix

Proof of Lemma 1: We note that the OLS estimator $\hat{\gamma}$ from the auxiliary regression in (3) can be written as:

$$\begin{aligned}\hat{\gamma} &= (\sum \tilde{z}_t \tilde{z}_t')^{-1} \sum \tilde{z}_t e_{\theta t} \\ &= (\sum z_t z_t')^{-1} \sum z_t \{\epsilon_{\theta t} - x_t'(\hat{\beta}_\theta - \beta_\theta)\} + o_p(\frac{1}{\sqrt{T}}) \\ &= (\sum z_t z_t')^{-1} \{\sum z_t \epsilon_{\theta t} - \sum z_t x_t' D_T^{-1} \sum x_t \psi_\theta(\epsilon_{\theta t})\} + o_p(\frac{1}{\sqrt{T}}),\end{aligned}$$

where $D_T = \frac{1}{T}\sum E[f_{\epsilon_{\theta t}}(0)^2 x_t x_t']$. The second equality uses $\frac{1}{T}\sum \tilde{z}_t\tilde{z}_t' = \frac{1}{T}\sum z_t z_t' + O_p(\frac{1}{\sqrt{T}})$ and Assumption 1 (iii). The last equality is obtained by applying Lemmas 8 and 9 and the proof of Theorem 4 of Komunjer (2005) to our linear quantile regression case where $a_t(y) = \frac{1}{\theta(1-\theta)} y$ and $q_t(x_t) = x_t'\beta_\theta$. That is, Assumption 1 satisfies all the conditions of Theorem 7.2 in Newey and McFadden (1994). Thus, using $\gamma = (\sum z_t z_t')^{-1} \sum z_t E[\epsilon_{\theta t}|x_t]$, we obtain that

$$\sqrt{T}(\hat{\gamma} - \gamma) = \begin{pmatrix} -D_T^{-1}(\frac{1}{T}\sum z_t x_t')'(\frac{1}{T}\sum z_t z_t')^{-1} \\ (\frac{1}{T}\sum z_t z_t')^{-1} \end{pmatrix}' \begin{pmatrix} \frac{1}{\sqrt{T}}\sum x_t \psi_\theta(\epsilon_{\theta t}) \\ \frac{1}{\sqrt{T}}\sum z_t \epsilon_{\theta t}^* \end{pmatrix} + o_p(1). \tag{11}$$

Since $\psi_\theta(\epsilon_{\theta t})$ is a measurable function of $(x_t, \epsilon_{\theta t})$, it is also mixing with the same mixing coefficients according to Theorem 14.1 of Davidson (1994). Therefore, under Assumption 1, we can apply the central limit theorem for mixing processes to the second term in (11) using Theorem 5.20 of White (2000) to obtain the following:

$$\begin{pmatrix} \frac{1}{\sqrt{T}}\sum x_t \psi_\theta(\epsilon_{\theta t}) \\ \frac{1}{\sqrt{T}}\sum z_t \epsilon_{\theta t}^* \end{pmatrix} \xrightarrow{d} W \sim N(0, Q),$$

where Q is given in Section 2. Using Theorem 3.49 of White (2000) and the point wise continuous mapping theorem, we can also easily show the probability limit of the first term in (11):

$$\begin{pmatrix} -D_T^{-1}(\frac{1}{T}\sum z_t x_t')'(\frac{1}{T}\sum z_t z_t')^{-1} \\ (\frac{1}{T}\sum z_t z_t')^{-1} \end{pmatrix} \xrightarrow{p} \Lambda = \begin{pmatrix} -D^{-1}M_{zx}'M_z^{-1} \\ M_z^{-1} \end{pmatrix}.$$

Hence, we obtain

$$\sqrt{T}(\hat{\gamma} - \gamma) \xrightarrow{d} N(0, \Lambda' Q \Lambda).$$

We note that $\Lambda' Q \Lambda$ can be rewritten as

$$\begin{aligned}\Lambda' Q \Lambda &= \theta(1-\theta) M_z^{-1} M_{zx} D^{-1} M_x D^{-1} M_{zx}' M_z^{-1} - M_z^{-1} M_{zx,\epsilon} D^{-1} M_{zx}' M_z^{-1} \\ &\quad - M_z^{-1} M_{zx} D^{-1} M_{zx,\epsilon}' M_z^{-1} + M_z^{-1} M_{z,\epsilon} M_z^{-1} \\ &= C_1 - C_2' - C_2 + M_z^{-1} M_{z,\epsilon} M_z^{-1},\end{aligned} \tag{12}$$

where the last identity defines C_1 and C_2. Comparing the conclusion of Lemma 1 and the above result in (12), the only step needed to complete the proof is to show that $A = C_1 - C_2 - C_2'$. For this, we first obtain

$$M_z^{-1} = \begin{pmatrix} P_0 & -M_x^{-1} M_{x\epsilon} P \\ -P M_{x\epsilon}' M_x^{-1} & P \end{pmatrix},$$

using the inverse formula of a partitioned matrix. Hence, we have

$$M_z^{-1}M_{zx} = \begin{pmatrix} P_0 M_x - M_x^{-1}M_{x\epsilon}PM'_{x\epsilon} \\ 0_{p,k} \end{pmatrix},$$

which implies

$$\begin{aligned} C_1 &= \theta(1-\theta)\begin{pmatrix} P_0 M_x - M_x^{-1}M_{x\epsilon}PM'_{x\epsilon} \\ 0_{p,k} \end{pmatrix} D^{-1}M_x D^{-1}\begin{pmatrix} P_0 M_x - M_x^{-1}M_{x\epsilon}PM'_{x\epsilon} \\ 0_{p,k} \end{pmatrix}' \\ &= \begin{pmatrix} A_1 & 0_{k,p} \\ 0_{p,k} & 0_{p,p} \end{pmatrix}. \end{aligned}$$

We also note that

$$\begin{aligned} M_z^{-1}M_{zx,\epsilon} &= \begin{pmatrix} P_0 & -M_x^{-1}M_{x\epsilon}P \\ -PM'_{x\epsilon}M_x^{-1} & P \end{pmatrix}\begin{pmatrix} M_{x,\epsilon} \\ M'_{x\epsilon,\epsilon} \end{pmatrix} \\ &= \begin{pmatrix} P_0 M_{x,\epsilon} - M_x^{-1}M_{x\epsilon,\epsilon}PM'_{x\epsilon,\epsilon} \\ -PM'_{x\epsilon}M_x^{-1}M_{x,\epsilon} + PM'_{x\epsilon,\epsilon} \end{pmatrix}, \end{aligned}$$

which in turn implies that

$$\begin{aligned} C_2 &= \begin{pmatrix} P_0 M_x - M_x^{-1}M_{x\epsilon}PM'_{x\epsilon} \\ 0_{p,k} \end{pmatrix} D^{-1}\begin{pmatrix} P_0 M_{x,\epsilon} - M_x^{-1}M_{x\epsilon,\epsilon}PM'_{x\epsilon,\epsilon} \\ -PM'_{x\epsilon}M_x^{-1}M_{x,\epsilon} + PM'_{x\epsilon,\epsilon} \end{pmatrix}' \\ &= \begin{pmatrix} A_2 & A_3 \\ 0_{p.k} & 0_{p.p} \end{pmatrix}. \end{aligned}$$

Therefore, $A = C_1 - C_2 - C_2'$, which completes the proof of Lemma 1. QED.

Proof of Theorem 1(i): We first note that LM_T can be rewritten as

$$LM_T = \frac{(\sum_{t=1}^T \tilde{e}_{\theta t}^2 - \sum_{t=1}^T \hat{v}_t^2)}{\sum_{t=1}^T \hat{v}_t^2/(T-p-k)}, \tag{13}$$

where $\hat{v}_t$ is as defined in (3), and $\tilde{e}_{\theta t}$ is the deviation of $e_{\theta t}$ from its sample mean. Hence, we have

$$\begin{aligned} LM_T &= (\sqrt{T}R_1\hat{\gamma})'(R_1 M_{Tz}^{-1}R_1')^{-1}(\sqrt{T}R_1\hat{\gamma})/s^2 + o_p(1) \\ &= (\sqrt{T}R_1(\hat{\gamma}-\gamma))'(R_1 M_{Tz}^{-1}R_1')^{-1}(\sqrt{T}R_1(\hat{\gamma}-\gamma))/s^2 \\ &\quad + T(R_1\gamma)'(R_1 M_{Tz}^{-1}R_1')^{-1}(R_1\gamma)/s^2 + 2\sqrt{T}(R_1\gamma)'(R_1 M_{Tz}^{-1}R_1')^{-1}(\sqrt{T}R_1(\hat{\gamma}-\gamma))/s^2 + o_p(1). \end{aligned} \tag{14}$$

where $s^2 = \frac{1}{T-k-p}\sum_{t=1}^T \hat{v}_t^2$. We can apply the law of large numbers for mixing sequence to obtain (i) $R_1 M_{Tz} R_1' \xrightarrow{p} R_1 M_z R_1'$ and (ii) $s^2 \xrightarrow{p} \sigma^2$. Hence, these results together with Lemma 1 imply that

$$\begin{aligned} (\sqrt{T}R_1(\hat{\gamma}-\gamma))'(R_1 M_{Tz}^{-1}R_1')^{-1}(\sqrt{T}R_1(\hat{\gamma}-\gamma))/s^2 &= O_p(1), \\ T(R_1\gamma)'(R_1 M_{Tz}^{-1}R_1')^{-1}(R_1\gamma)/s^2 &= O_p(T), \\ \sqrt{T}(R_1\gamma)(R_1 M_{Tz}^{-1}R_1')^{-1}(\sqrt{T}R_1(\hat{\gamma}-\gamma))/s^2 &= O_p(\sqrt{T}), \end{aligned}$$

which delivers the desired result. QED.

Proof of Theorem 1(ii): Under homoscedasticity, $\gamma_0 = (\mu_{\epsilon_\theta}, 0'_{k-1,1})'$ so that $R_1\gamma = 0_{p+k-1,1}$. Hence, the second and third terms in (14) disappear. Consequently, by applying Lemma 1,

$$\sqrt{T}R_1(\hat{\gamma} - \gamma) \Rightarrow B,$$

where $B \sim N\left(0_{k+p-1,1}, R_1(A + M_z^{-1}M_{z,\epsilon}M_z^{-1})R_1'\right)$ and $R_1 = (0_{k+p-1,1}, I_{k+p-1})$. Hence, the proof is now completed. QED.

Proof of Theorems 2 & 3: We first consider the heteroscedasticity case in which the relevant statistic is given by

$$QF_T^H = (\sqrt{T}R_2\hat{\gamma})(R_2\hat{V}R_2')^{-1}(\sqrt{T}R_2\hat{\gamma})',$$

where $\hat{V} = \tilde{M}_{Tz}^{-1}M_{Tz,\epsilon}\tilde{M}_{Tz}^{-1}$, and $M_{Tz,\epsilon} = \frac{1}{T}\sum_{t=1}^{T} e_{\theta t}^2 \tilde{z}_t\tilde{z}_t'$. Since $R_2\gamma = 0_{p,1}$ and $R_2AR_2' = 0_{p,p}$, we can apply Lemma 1 to obtain that $R_2\hat{\gamma} = R_2(\hat{\gamma} - \gamma) \xrightarrow{d} N(0_{p,1}, R_2M_z^{-1}M_{z,\epsilon}M_z^{-1}R_2')$. Hence, we have that $QF_T^H \xrightarrow{d} \mathcal{X}_p^2$. If $\epsilon_{\theta t}$ is homoscedastic, then the proof is analogous to the heteroscedasticity case, except $(M_{Tz}^{-1}M_{Tz,\epsilon}M_{Tz}^{-1}) \xrightarrow{p} M_z^{-1}/\sigma^2$ because $M_{z,\epsilon} = \sigma^2 M_z$ under homoscedasticity, which again implies that $QF_T \xrightarrow{d} \mathcal{X}_p^2$. QED.

Table 1. Rejection frequencies (%) of the conventional LM test with $p = 2$ (size)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	5.7	29.5	19.6	10.2	7.7	10.8	20.5	29.9
300	5.1	34.1	19.0	9.9	7.0	8.5	18.9	31.1
500	5.2	36.4	23.2	10.9	7.4	10.3	21.6	33.6
1,000	4.5	33.2	19.2	9.9	6.3	9.4	19.3	33.1
3,000	4.6	31.2	21.8	8.1	6.7	8.7	18.7	31.1
5,000	5.6	30.2	20.5	10.0	8.3	10.1	21.3	32.6
				$t(5)$				
50	4.2	35.8	21.7	7.6	6.1	7.3	20.1	37.6
300	4.3	37.9	21.9	7.9	6.3	8.2	21.6	38.4
500	5.0	40.7	23.1	9.2	7.2	9.8	21.8	44.6
1,000	5.3	38.7	20.4	8.7	7.7	10.3	21.5	39.4
3,000	5.4	41.1	22.6	9.3	7.4	9.0	22.0	42.3
5,000	5.2	40.8	23.4	9.7	6.9	10.3	22.3	43.3
				$LN(1,0.4)$				
50	5.0	14.9	12.4	8.4	7.3	10.4	31.8	49.6
300	4.5	14.3	11.0	7.9	7.2	9.9	32.3	51.3
500	4.8	15.6	12.3	8.7	7.3	12.4	36.7	53.7
1,000	4.9	15.5	11.8	8.6	7.2	11.2	33.9	55.8
3,000	4.8	13.3	11.4	9.3	7.9	10.6	34.0	53.8
5,000	5.4	12.5	11.8	9.0	8.2	12.4	36.5	52.0

Table 2. Rejection frequencies (%) of the conventional LM test (size)

	OLS				Quantile regression ($\theta = 0.9$)			
T	$p = 1$	$p = 2$	$p = 4$	$p = 12$	$p = 1$	$p = 2$	$p = 4$	$p = 12$
				$N(0,1)$				
50	5.4	5.7	5.3	1.5	28.5	20.5	14.0	2.2
300	5.5	5.1	4.8	3.9	24.8	18.9	15.7	8.6
500	4.8	5.2	5.8	4.6	28.2	21.6	16.4	9.6
1,000	5.2	4.5	3.9	4.9	24.7	19.3	14.6	11.4
3,000	4.5	4.6	4.2	4.3	24.9	18.7	14.4	9.1
5,000	4.7	5.6	6.3	3.9	27.4	21.3	17.0	10.1
				$t(5)$				
50	4.5	4.2	3.3	0.7	28.1	20.1	11.9	2.4
300	4.8	4.3	4.3	3.5	29.5	21.6	16.2	8.4
500	6.1	5.0	5.3	4.9	28.6	21.8	17.5	10.7
1,000	5.3	5.3	4.7	4.5	27.7	21.5	15.0	11.4
3,000	3.7	5.4	4.7	3.9	27.7	21.9	16.9	11.2
5,000	4.4	5.2	3.7	4.4	29.0	22.3	16.2	12.0
				$LN(1,0.4)$				
50	4.8	5.0	4.8	1.1	41.3	31.8	22.5	4.3
300	4.7	4.5	5.1	2.4	40.1	32.3	25.7	16.2
500	4.6	4.8	4.3	4.4	43.2	36.7	27.9	18.8
1,000	4.9	4.9	4.9	4.9	43.1	33.9	26.6	18.3
3,000	5.0	4.8	4.7	4.2	40.1	34.0	27.0	17.6
5,000	4.0	5.4	5.3	4.2	40.4	36.5	30.0	18.6

Table 3. Rejection frequencies (%) of the QF test with $p = 2$ (size)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	5.4	5.3	5.8	5.2	5.2	5.3	4.5	5.3
300	4.9	4.5	4.8	5.0	4.8	4.7	5.3	4.9
500	5.2	5.3	5.4	5.2	5.3	5.1	5.3	5.1
1,000	4.5	4.8	4.8	4.4	4.5	4.5	4.6	4.8
3,000	4.6	4.6	4.6	4.6	4.5	4.6	4.6	4.7
5,000	5.6	5.8	5.7	5.7	5.5	5.6	5.4	5.5
				$t(5)$				
50	4.0	3.5	4.3	3.6	3.9	4.0	4.1	3.7
300	4.3	4.7	4.5	4.4	4.4	4.3	4.7	4.7
500	5.0	5.0	5.1	4.5	4.8	5.2	4.9	5.2
1,000	5.3	5.4	5.3	5.3	5.3	5.3	5.1	4.8
3,000	5.4	5.3	5.2	5.1	5.5	5.3	5.3	5.3
5,000	5.2	5.2	5.2	5.0	5.2	5.2	5.1	5.3
				$LN(1,0.4)$				
50	5.0	4.5	4.7	4.5	4.7	4.7	4.7	4.0
300	4.3	4.3	4.2	4.2	4.6	4.5	4.3	4.4
500	4.8	5.0	5.0	4.8	4.7	4.9	5.2	4.5
1,000	4.9	5.0	5.3	5.2	5.0	4.9	5.1	4.9
3,000	4.8	5.1	4.9	5.0	4.9	5.0	4.9	4.7
5,000	5.4	5.4	5.4	5.4	5.3	5.2	5.2	5.2

Table 4. Rejection frequencies (%) of the QF test (size)

	OLS				Quantile regression ($\theta = 0.9$)			
T	$p = 1$	$p = 2$	$p = 4$	$p = 12$	$p = 1$	$p = 2$	$p = 4$	$p = 12$
				$N(0,1)$				
50	5.0	5.4	5.1	2.2	4.8	4.5	4.5	1.9
300	5.4	4.9	4.8	3.9	5.2	5.3	5.1	4.0
500	4.8	5.2	5.8	4.6	5.0	5.3	5.7	4.5
1,000	5.2	4.5	3.9	4.9	5.2	4.6	3.7	5.1
3,000	4.5	4.6	4.2	4.3	4.4	4.6	4.5	4.4
5,000	4.7	5.6	6.3	3.9	4.8	5.4	6.3	3.8
				$t(5)$				
50	3.8	4.0	3.2	2.4	3.8	4.1	3.4	2.2
300	4.8	4.3	4.3	3.6	4.8	4.7	4.4	3.5
500	6.1	5.0	5.2	5.0	6.5	4.9	5.2	4.6
1,000	5.3	5.3	4.6	4.6	5.1	5.1	4.4	4.5
3,000	3.7	5.4	4.7	3.9	3.8	5.3	4.7	4.0
5,000	4.4	5.2	3.7	4.4	4.4	5.1	3.8	4.4
				$LN(1,0.4)$				
50	4.3	5.0	4.6	2.0	3.8	4.7	3.5	2.0
300	4.6	4.3	5.1	2.6	4.9	4.3	4.9	3.0
500	4.4	4.8	4.3	4.5	4.7	5.2	4.3	4.2
1,000	4.8	4.9	4.9	4.9	4.9	5.1	4.6	4.6
3,000	5.0	4.8	4.7	4.2	5.1	4.9	5.1	4.5
5,000	4.0	5.4	5.3	4.2	3.9	5.2	5.3	4.0

Table 5. Rejection frequencies (%) of the QR-LM test (size)

T	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$			
50	6.2	6.0	6.5	7.1	6.5	6.1	5.9
300	5.3	5.6	5.3	4.9	5.1	5.5	3.8
500	5.9	4.7	5.5	4.5	7.0	5.1	5.6
1,000	5.9	5.1	5.1	6.0	4.7	5.4	5.4
3,000	6.6	4.3	4.2	3.6	4.4	4.8	5.7
5,000	4.8	4.3	5.3	5.1	4.9	4.9	4.9
				$t(5)$			
50	7.6	5.3	5.7	5.2	6.3	6.7	7.6
300	4.7	5.3	4.3	4.8	6.0	4.2	5.0
500	4.0	5.5	5.9	4.6	5.4	4.1	4.4
1,000	5.2	4.4	5.2	5.3	5.3	5.7	6.4
3,000	5.5	4.8	5.0	4.7	5.1	4.8	6.1
5,000	5.9	4.6	5.9	4.9	5.6	5.2	4.0
				$LN(1,0.4)$			
50	8.1	7.0	6.6	7.3	6.6	5.0	4.2
300	6.3	6.4	5.3	5.3	5.1	4.9	3.5
500	6.0	4.9	5.2	5.5	6.6	5.1	5.7
1,000	5.2	6.3	4.1	5.3	4.2	4.9	5.1
3,000	6.4	4.8	4.1	3.8	4.6	4.7	6.4
5,000	4.9	4.3	5.4	5.7	4.9	5.2	5.0

Table 6. Rejection frequencies (%) of the QR-LM test with $\theta = 0.9$ (size)

T	$p = 1$	$p = 2$	$p = 4$	$p = 12$
		$N(0,1)$		
50	6.7	6.1	5.1	2.6
300	4.4	5.5	5.2	4.1
500	5.0	5.1	5.2	3.3
1,000	4.4	5.4	5.4	5.2
3,000	6.3	4.8	4.4	4.4
5,000	4.0	4.9	5.5	5.8
		$t(5)$		
50	7.0	6.7	5.6	3.1
300	5.0	4.2	2.9	3.9
500	4.3	4.1	5.5	4.6
1,000	6.3	5.7	5.8	4.7
3,000	4.6	4.8	6.9	4.8
5,000	5.6	5.2	4.8	4.9
		$LN(1,0.4)$		
50	5.5	5.0	4.4	2.1
300	4.5	4.9	5.3	4.1
500	5.0	5.1	4.8	4.2
1,000	5.2	4.9	5.2	5.1
3,000	5.6	4.7	4.8	4.9
5,000	4.6	5.2	5.8	5.1

Table 7. Rejection frequencies(%) of the QF test with $p = 2$ & $\rho = 0.4$ (power)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	58.3	56.5	57.4	58.7	58.2	58.2	56.3	56.4
100	94.2	93.1	94.2	93.8	93.9	93.5	93.1	93.1
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(5)$				
50	61.0	55.8	58.0	60.0	60.3	59.9	58.4	55.3
100	94.0	91.7	93.1	93.4	93.9	93.3	93.6	92.1
200	99.7	99.7	99.7	99.7	99.7	99.7	99.7	99.7
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$LN(1, 0.4)$				
50	56.9	57.3	58.1	57.9	57.3	55.6	53.6	50.4
100	93.9	93.9	93.8	93.7	93.9	93.1	92.4	89.4
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 8. Rejection frequencies(%) of the QF test with $p = 2$ & $\rho = 0.8$ (power)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	99.8	99.6	99.8	99.8	99.8	99.9	99.7	99.5
100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(5)$				
50	99.5	99.6	99.4	99.5	99.4	99.5	99.4	99.3
100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$LN(1, 0.4)$				
50	99.9	99.7	99.9	99.8	99.8	99.9	99.7	99.4
100	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 9. Rejection frequencies(%) of the QR-LM test with $p = 2$ & $\rho = 0.4$ (power)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	60.2	13.1	19.8	36.0	40.2	34.0	18.4	16.1
100	94.3	28.9	43.2	70.5	77.1	70.6	47.5	32.2
200	100.0	62.9	82.2	97.5	99.0	96.2	80.7	61.1
300	100.0	83.5	95.8	100.0	100.0	99.9	95.6	81.9
				$t(5)$				
50	62.5	12.6	20.5	39.8	51.8	39.6	21.2	13.6
100	94.3	29.1	44.6	76.7	87.2	76.8	47.8	31.2
200	99.7	51.8	80.1	98.6	99.7	98.3	80.6	51.5
300	100.0	71.4	94.2	99.8	100.0	99.7	93.9	71.0
				$LN(1,0.4)$				
50	58.3	8.8	16.3	44.0	51.1	36.4	21.3	17.2
100	94.3	33.1	63.4	88.3	89.0	68.5	41.5	30.4
200	100.0	86.7	98.1	99.8	99.7	96.2	69.7	47.7
300	100.0	99.2	100.0	100.0	100.0	99.7	86.5	60.7

Table 10. Rejection frequencies(%) of the QR-LM test with $p = 2$ & $\rho = 0.8$ (power)

T	OLS	$\theta = 0.05$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 0.9$	$\theta = 0.95$
				$N(0,1)$				
50	99.8	39.7	65.2	94.5	97.5	92.7	66.4	42.3
100	100.0	89.9	98.7	99.9	100.0	100.0	98.8	89.7
200	100.0	99.9	100.0	100.0	100.0	100.0	100.0	99.8
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$t(5)$				
50	99.5	40.0	64.8	94.0	98.3	93.6	66.6	40.7
100	100.0	86.5	98.0	100.0	100.0	99.9	98.3	87.2
200	100.0	99.4	100.0	100.0	100.0	100.0	100.0	100.0
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
				$LN(1,0.4)$				
50	99.9	37.9	66.0	97.2	98.3	93.9	66.4	45.0
100	100.0	92.4	99.7	100.0	100.0	100.0	97.5	85.1
200	100.0	100.0	100.0	100.0	100.0	100.0	100.0	98.9
300	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 11. Quantile regression results of the Fama-French model (72 observations)

	OLS	$\theta = 0.05$	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 0.90$	$\theta = 0.95$
Quantile coefficients								
β_1 (market)	1.1148	1.0209	1.2454	1.1325	0.9428	1.1361	0.9949	0.9155
β_2 (SMB)	-0.2575	-0.8273	-0.7291	-0.7272	-0.3897	0.0559	0.1677	0.6085
β_3 (HML)	-0.1605	0.3592	0.1245	-0.0410	0.2436	-0.2948	-0.3536	-0.7212
β_0 (intercept)	0.3770	-6.2329	-4.9197	-3.1117	0.0297	3.9406	5.3457	8.7135
Test results								
LM stat.	1.0913	9.1255	6.4235	3.9166	5.3538	1.9412	3.2296	13.1499
p-value	0.2962	0.0025	0.0113	0.0478	0.0207	0.1635	0.0723	0.0003
QF stat.	1.0084	0.8619	1.3569	1.1463	0.6847	0.9445	0.6572	0.3884
p-value	0.3190	0.3566	0.2483	0.2882	0.4110	0.3347	0.4205	0.5353