연세대학교 상경대학
경제연구소
Economic Research Institute
Yonsei Universit

서울시 서대문구 연세로 50
50 Yonsei-ro, Seodaemun-gu, Seoul, Korea
TEL: (+82-2) 2123-4065 FAX: (+82-2) 364-9149
E-mail: yeri4065@yonsei.ac.kr
http://yeri.yonsei.ac.kr/new

Competitive Equilibrium and Singleton Cores in Generalized Matching Problems

Jaeok Park
Yonsei University

October 2015
2015RWP-85

Economic Research Institute
Yonsei University

Competitive Equilibrium and Singleton Cores in Generalized Matching Problems

Jaeok Park*

September 29, 2015

Abstract

We study competitive equilibria in generalized matching problems. We show that, if there is a competitive matching, then it is unique and the core is a singleton consisting of the competitive matching. That is, a singleton core is necessary for the existence of competitive equilibria. We also show that a competitive matching exists if and only if the matching produced by the top trading cycles algorithm is feasible, in which case it is the unique competitive matching. Hence, we can use the top trading cycles algorithm to test whether a competitive equilibrium exists and to construct a competitive equilibrium if one exists. Lastly, in the context of bilateral matching problems, we compare the condition for the existence of competitive matchings with existing sufficient conditions for the existence or uniqueness of stable matchings and show that it is weaker than the existing conditions.

Keywords: matching, competitive equilibrium, core, top trading cycles algorithm.

JEL Classification: C78, D51.

1 Introduction

Competitive equilibrium and the core are two of the most important concepts in economics. The existence and uniqueness of competitive equilibrium and core allocations and the relationship between the two have been studied extensively in various settings including matching markets. In their pioneering work, Gale and Shapley (1962) introduce two classes of matching problems called marriage problems and roommate problems and study the core (or stable matchings) of these problems. They show that any marriage problem has a nonempty core while a roommate problem may have an empty core. Moulin (1995, Ch.

*School of Economics, Yonsei University, 50 Yonsei-ro, Seodaemun-gu, Seoul 03722, Korea (e-mail: jaeok.park@yonsei.ac.kr, phone: +82-2-2123-6572, fax: +82-2-2123-8638)

3.3) applies the concept of competitive equilibrium to marriage problems and provide two examples to demonstrate that a (unique) core matching in a marriage problem may or may not be interpreted as a competitive allocation. Shapley and Scarf (1974) consider another class of matching problems called housing markets and show that the (weak) core is nonempty and that there is a competitive allocation in the core. Later Roth and Postlewaite (1977) sharpen Shapley and Scarf's (1974) result by showing that, when agents have strict preferences, the (strong) core is a singleton which consists of the unique competitive allocation.

To sum up the discussion in the opening paragraph, while a tight connection between competitive equilibria and the core of housing markets was established several decades ago, it is yet to be seen when there exists a competitive equilibrium and how competitive allocations are related to core matchings in marriage and roommate problems. This paper aims to fill this gap in the literature by investigating these questions. To this end, we study competitive equilibria in generalized matching problems, which are introduced by Sönmez (1996) and include marriage and roommate problems and housing markets as special cases. By taking this approach, we can see clearly how the existing result on competitive equilibria and the core of housing markets in Roth and Postlewaite (1977) generalizes to a broader context and what features of housing markets are crucial in obtaining their result.

Talking about competitive equilibria of marriage or roommate problems may sound unnatural or inappropriate, as we need to price people and let them purchase each other (see Roth, 2007, for related arguments). However, as pointed out in Moulin (1995), we can interpret marriage and roommate problems as exchange economies in which each individual owns an indivisible good and trades are restricted to be bilateral. With this interpretation, a price mechanism is no less applicable to marriage and roommate problems than to housing markets. Studying competitive equilibria of matching problems is important from both a theoretical and practical point of view. The core can be considered as the set of outcomes that can arise from cooperative agreement, and since Edgeworth (1881) it has been examined whether the same outcomes can be obtained with decentralized price-taking behavior. Hence, it is theoretically meaningful to investigate whether the equivalence holds in widely-studied matching models. If we find that a competitive allocation exists and it coincides with a core outcome, it has practical implications. We can endow agents with fiat money or tokens and let them trade their goods to achieve a cooperative outcome in a decentralized fashion. Furthermore, we can attempt to design a tâtonnement mechanism to achieve the same outcome as centralized (preference-reporting) mechanisms, and such a tâtonnement mechanism may provide an informationally efficient process to obtain a core allocation.

In a generalized matching problem, each agent is endowed with one indivisible good and has strict preferences on a subset of the goods held by all agents. A matching assigns each agent one of the goods he values in a way that no good is allocated to more than one agent, and the set of feasible matchings is specified exogenously (for example, in marriage and roommate problems, only bilateral matchings are feasible). A generalized matching problem can be interpreted as an exchange economy where each agent has a unit supply of his good and demands one of the available goods, while feasible allocations or trades are described by feasible matchings. Using this interpretation, we can define competitive equilibria of generalized matching problems in a natural way. At a competitive equilibrium, each agent's endowment is priced, and each agent is assigned the most preferred good among those he can afford given the prices. The market clearing conditions are taken care of by requiring the allocation to be a feasible matching. We say that a matching is competitive if it constitutes a competitive equilibrium together with a supporting price vector.

We first show that every competitive matching is in the core (Proposition 1), which is a result that holds in general. As shown in Moulin (1995), a marriage problem may not have a competitive matching. Thus, competitive equilibrium is more restrictive than the core. We next show that, if there is a competitive matching, the core is a singleton consisting of the competitive matching (Theorem 1). That is, a singleton core is necessary for the existence of competitive equilibria. Also, provided that a competitive matching exists, we have equivalence between the set of competitive matchings and the core as a singleton. We then turn to the existence problem and prove that a competitive equilibrium exists if and only if Gale's top trading cycles algorithm (Shapley and Scarf, 1974) yields a feasible matching (Theorem 2). Thus, the top trading cycles algorithm can be used to test whether a competitive equilibrium exists and to construct a competitive matching and a supporting price vector if one exists. This result explains the existence and uniqueness of competitive and core matchings in housing markets as shown in Roth and Postlewaite (1977), because every matching is feasible in housing markets (Corollary 1). In contrast, in roommate and marriage problems where only bilateral matchings are allowed, a competitive matching exists if and only if every top trading cycle formed at each step of the top trading cycles algorithm has length 1 or 2 (Corollary 2). We call this condition iterative α–reducibility as it weakens the concept of α–reducibility introduced in Alcalde (1995), and it can be considered as a sufficient condition for the uniqueness of stable matchings in roommate and marriage problems. We compare it with other existing sufficient conditions in the literature and show that it is weaker than the existing ones (Propositions 2 and 3).

Singleton cores have played an important role in the matching literature. In particular, singleton cores have been related to agents' incentives in centralized matching mechanisms. In marriage problems, where there can be multiple stable matchings, there is no strategy-proof mechanism that selects a stable matching (Roth, 1982a). In contrast, in housing markets, where the core is a singleton, the core mechanism is strategy-proof (Roth, 1982b) and it is the only mechanism that is Pareto-efficient, individually rational, and strategy-proof (Ma, 1994). Sönmez (1999) considers allocation problems of indivisible goods that include marriage problems and housing markets as subclasses, and he shows that singleton cores are necessary to have a mechanism that is Pareto-efficient, individually rational, and strategy-proof.[1] Notice the parallel between Sönmez (1999) and this paper. We have the negative result of the possible nonexistence of competitive matchings in marriage problems and the positive result of the existence and uniqueness of competitive matchings in housing markets. We consider generalized matching problems that include marriage problems and housing markets and find that singleton cores are necessary for the existence of competitive matchings. Hence, this paper contributes to the literature by pointing out that singleton cores are crucial not only for centralized matching mechanisms to be nonmanipulable but also for decentralized price mechanisms to function in matching markets. There are other studies that relate singleton cores to strategic behavior in centralized matching mechanisms. Ma (2002) considers college admissions problems and shows that the core is a singleton for the reported preference profile at a Nash equilibrium in truncations in a stable mechanism. Ehlers and Massó (2007) consider marriage problems with incomplete information and show that truth-telling is an ordinal Bayesian Nash equilibrium in a stable mechanism if and only if the core is a singleton for each preference profile in the support of the common belief.

In addition, the literature has provided empirical and theoretical findings that singleton cores are ubiquitous. There is strong evidence that the core is small and often a singleton in real-world matching problems such as hospital-resident matching (Roth and Peranson, 1999) and school choice (Pathak and Sönmez, 2008).[2] Ashlagi et al. (2015) provide theoretical support for singleton cores by considering marriage problems with unequal numbers of men and women and random heterogeneous preferences. They show that as the number of agents becomes larger, unbalanced marriage problems have a unique stable matching with high probability. Since singleton cores are only necessary but not sufficient for the existence of competitive equilibria, matching problems with a singleton core may not have a competitive matching. Hence, it still remains to be seen both empirically and theoretically how likely various matching problems have a competitive matching. In an effort to address this issue, we present numerical results on marriage problems with random preferences as considered in Ashlagi et al. (2015). Our results show that, as there are more agents on

[1] Sönmez (1999) allows indifference between two distinct allocations, and in this case "essentially" singleton cores are necessary, where all allocations in the core are Pareto indifferent.

[2] Using data for five years 1991–1994 and 1996 for the thoracic surgery market, Roth and Peranson (1999) report that there are two stable matchings in 1992 and 1993 and there is only one stable matching in 1991, 1994, and 1996. Similarly, using data for two school years 2005–2006 and 2006–2007 for Boston Public School student admissions, Pathak and Sönmez (2008) find that there is only one stable matching in either year at grade K2 and there are two stable matchings in either year at grade 6.

one or both sides, it becomes less likely for a competitive matching to exist, and with 10 or more agents on each side, it is rarely the case that a competitive matching exists. In our model, each agent owns his personalized good, and thus the number of goods is equal to the number of agents. When there are many agents and thus many goods, bilateral trades are not sufficient to realize all gains from trade, prohibiting the existence of competitive matchings.

Our work is related to studies on the core and competitive equilibria in matching markets or exchange economies with indivisibilities. Most existing work assumes that money is available as a commodity, in contrast to our model where there is no money commodity. Quinzii (1984) adds money to housing markets assuming that money enters agents' utility functions in a general way. She shows that the core is nonempty and coincides with the set of competitive allocations.[3] She also considers marriage problems with money and shows that the core is nonempty while a competitive equilibrium may not exist. Her results suggest that the main findings without money remain valid even when money is introduced. Shapley and Shubik (1971) study the assignment game in which buyers and sellers are matched to create surplus. Assuming quasilinear preferences in money, they show that the core is nonempty and coincides with the set of competitive allocations. Sotomayor (2007) extends the assignment game by considering many-to-many matching with additively separable preferences, and she compares setwise-stable payoffs with competitive equilibrium payoffs. The key feature of the assignment game that creates the difference from marriage problems is that agents are divided into two groups of buyers and sellers whereas in marriage problems agents play the roles of a buyer and a seller at the same time. In other words, in the assignment game agents in only one group, sellers, have endowments which are priced, while in marriage problems all agents on both sides have endowments. Crawford and Knoer (1981) and Kelso and Crawford (1982) extend college admissions problems by adding money in a quasilinear way with separable and substitutable preferences, respectively. They allow payment (salary) to depend on the pair of agents who are matched and show that the core is nonempty and coincides with the set of competitive allocations. Recently, Hatfield et al. (2013) study a bilateral trading network where trades are priced and agents have quasilinear preferences in money. They show that, under fully substitutable preferences, the set of stable outcomes (which are in the core) is essentially equivalent to the set of competitive equilibria. The above three papers are different from ours in that they allow match-specific, personalized prices whereas we focus on uniform prices (which is also the case in the assignment game). Bikhchandani and Mamer (1997) analyze exchange economies with indivisibilities where agents can consume multiple goods and have quasilin-

[3] Moulin (1995) shows that, if some agents own more than one indivisible good, the set of competitive allocations may be a proper subset of the core.

ear preferences in money. They present a necessary and sufficient condition for the existence of competitive equilibria and link their model to that of Kelso and Crawford (1982).

The rest of this paper is organized as follows. In Section 2, we introduce generalized matching problems and two solution concepts, the core and competitive equilibrium. In Section 3, we study the relationship between competitive equilibria and the core as well as the existence and uniqueness of competitive matchings. In Section 4, we compare the necessary and sufficient condition for the existence of competitive matchings with other sufficient conditions for the existence or uniqueness of stable matchings in the context of bilateral matching problems. In Section 5, we conclude.

2 Generalized Matching Problems

Our model follows the model of generalized matching problems, formulated by Sönmez (1996). Let N be a finite set of agents. For each $i \in N$, $S_i \subseteq N$ denotes the set of possible assignments for agent i, and R_i denotes agent i's preference relation. We assume that $i \in S_i$ for all $i \in N$, that is, each agent can be assigned his endowment. We assume that the preference relation R_i of each agent i is a linear order (i.e., transitive, antisymmetric, and total binary relation) on S_i. Antisymmetry of R_i excludes indifference between distinct assignments. Let P_i denote the strict relation associated with R_i, for all $i \in N$. A *(generalized) matching problem* is described by the triple $G = (N, S, R)$, where $S = (S_i)_{i \in N}$ and $R = (R_i)_{i \in N}$.

A *matching* for a matching problem $G = (N, S, R)$ is a bijection μ from N to itself such that $\mu(i) \in S_i$ for all $i \in N$. That is, a matching allocates to each agent one of his possible assignments in a way that no agent's endowment is allocated to more than one agent. For all $i \in N$, we refer to $\mu(i)$ as the assignment of i at μ. Let $\mathscr{M}$ be the set of all matchings. Let μ^I be the identity matching such that $\mu^I(i) = i$ for all $i \in N$. We specify a subset $\mathscr{M}^f$ of $\mathscr{M}$ as the set of *feasible matchings*. We require that $\mu^I \in \mathscr{M}^f$.

We study two solution concepts of matching problems, the core and competitive equilibrium. We first define the core. Given a matching $\mu \in \mathscr{M}^f$, suppose that there exist a coalition $T \subseteq N$ and another matching $\mu' \in \mathscr{M}^f$ such that (i) $\mu'(i) \in T$ for all $i \in T$, (ii) $\mu'(i) R_i \mu(i)$ for all $i \in T$, and (iii) $\mu'(i) P_i \mu(i)$ for some $i \in T$. In this case, we say that μ' *(weakly) dominates* μ via T and that T *blocks* μ. A matching $\mu \in \mathscr{M}^f$ is in the *(strong) core* of the matching problem $G = (N, S, R)$ if it is not dominated by any matching.

We now define competitive equilibria. In the competitive framework for matching problems, agents' endowments are priced and each agent chooses the most preferred assignment among affordable ones. Let $p \in \mathbb{R}^N_+$ be a price vector, where p_i denotes the price of agent i's endowment for all $i \in N$. An agent i's budget set given the price vector p

is the set of assignments that are possible and affordable for agent i and is denoted by $B_i(p) = \{j \in S_i : p_j \leq p_i\}$. Note that each agent can afford his own endowment regardless of the prices, i.e., $i \in B_i(p)$ for all $p \in \mathbb{R}^N_+$, for all $i \in N$. An agent i's demand set given the price vector p consists of optimal assignments in his budget set and is denoted by $D_i(p)$. That is, $D_i(p)$ is the set of maximal elements of $B_i(p)$ with respect to R_i. Since $B_i(p)$ is nonempty and R_i is a linear order, $D_i(p)$ is a singleton for every p. A pair $(p, \mu) \in \mathbb{R}^N_+ \times \mathscr{M}^f$ of a price vector and a matching is a *competitive equilibrium* of the matching problem $G = (N, S, R)$ if $\mu(i) \in D_i(p)$ for all $i \in N$. Note that the market clearing (or feasibility) conditions are taken care of by requiring μ to be a feasible matching. A price vector p *supports* a matching μ if (p, μ) is a competitive equilibrium. A matching μ is *competitive* if there exists a price vector p that supports μ.

As noted by Sönmez (1996), subclasses of generalized matching problems include *housing markets* (Shapley and Scarf, 1974) and *roommate and marriage problems* (Gale and Shapley, 1962). If $S_i = N$ for all $i \in N$ and $\mathscr{M}^f = \mathscr{M}$, we have a housing market. We say that a matching $\mu \in \mathscr{M}$ is bilateral if $\mu(\mu(i)) = i$ for all $i \in N$. That is, at a bilateral matching, if agent i is assigned agent j's endowment, agent j should be assigned agent i's endowment. In other words, at a bilateral matching, agents are matched into pairs or remain single. Let $\mathscr{M}^b$ be the set of all bilateral matchings. We refer to a matching problem with $\mathscr{M}^f = \mathscr{M}^b$ as a bilateral matching problem. Roommate and marriage problems are special cases of bilateral matching problems. If $S_i = N$ for all $i \in N$ and $\mathscr{M}^f = \mathscr{M}^b$, we have a roommate problem. If N can be partitioned into two nonempty disjoint sets M and W (sets of men and women), $S_m = W \cup \{m\}$ for all $m \in M$, $S_w = M \cup \{w\}$ for all $w \in W$, and $\mathscr{M}^f = \mathscr{M}^b$, we have a marriage problem.

In bilateral matching problems, it is common to use stability as a solution concept. Agent j is *acceptable* to agent i if agent i prefers agent j's endowment to his own (i.e., $jP_i i$). An agent i *blocks* the matching μ if he is matched with an unacceptable agent (i.e., $iP_i\mu(i)$). A pair (i, j) of distinct agents *blocks* the matching μ if agents i and j prefer each other's endowment to their assignments at μ (i.e., $jP_i\mu(i)$ and $iP_j\mu(j)$). A matching is *stable* if it is not blocked by any agent or any pair. The core of a bilateral matching problem coincides with the set of stable matchings, as any blocking coalition should involve a blocking agent or pair.

3 Properties of Competitive Matchings

In this section, we study the properties of competitive equilibria of generalized matching problems. We consider a fixed matching problem $G = (N, S, R)$ in this section, and thus we will not refer to it in our results. Given a matching $\mu \in \mathscr{M}$, a subset $C \subseteq N$ of agents is called a *trading cycle at* μ if (i) $\mu(i) \in C$ for all $i \in C$, and (ii) there is no proper subset C' of C with the property (i). Since μ is a bijection, we can express a trading cycle C as $\{i, \mu(i), \mu^2(i), \ldots, \mu^{K-1}(i)\}$ for any $i \in C$, where $K = |C|$, $\mu^1 = \mu$, and $\mu^{k+1} = \mu \circ \mu^k$ for all $k = 1, 2, \ldots$. Since the sequence $(i, \mu(i), \mu^2(i), \ldots, \mu^{K-1}(i))$ forms a cycle (i.e., $\mu^K(i) = i$) where i obtains the endowment of $i' = \mu(i)$, i' obtains the endowment of $i'' = \mu^2(i)$, and so on, we adopt the name trading cycles and sometimes use the sequence to represent the trading cycle. Note that, if $\mu(i) = i$, the singleton set $\{i\}$ forms a trading cycle. A matching $\mu \in \mathscr{M}$ partitions uniquely the set of agents into trading cycles at μ.[4] Hence, given a matching μ, each agent belongs to exactly one trading cycle at μ, and we denote the trading cycle at μ containing i by C_i^μ. In the following lemma, we provide simple observations on price vectors supporting a given matching.

Lemma 1. *Let $\mu \in \mathscr{M}^f$ be a feasible matching. Then a price vector $p \in \mathbb{R}_+^N$ supports μ if and only if for all $i \in N$, (i) $p_i = p_j$ for all $j \in C_i^\mu$ and (ii) $p_i < p_j$ for all $j \in S_i$ such that $jP_i\mu(i)$.*

Proof. ($\Rightarrow$) Suppose that p supports μ. Choose any $i \in N$. We can express C_i^μ as $\{i_0, i_1, i_2, \ldots, i_{K-1}\}$ where $K = |C_i^\mu|$, $i_0 = i$, $i_k = \mu^k(i)$ for all $k = 1, \ldots, K-1$, and $\mu^K(i) = i$. Since $i_{k+1} = \mu(i_k) \in B_{i_k}(p)$, we have $p_{i_k} \geq p_{i_{k+1}}$ for all $k = 0, \ldots, K-1$. Since $i_0 = i_K$, this implies $p_{i_0} = p_{i_1} = \cdots = p_{i_{K-1}}$. Hence, $p_i = p_j$ for all $j \in C_i^\mu$. Choose any $j \in S_i$ such that $jP_i\mu(i)$. Suppose that $p_j \leq p_i$. Then $j \in B_i(p)$, and $\mu(i)$ is not an optimal assignment in agent i's budget set, a contradiction. Hence, $p_j > p_i$.

($\Leftarrow$) Suppose that, for all $i \in N$, (i) $p_i = p_j$ for all $j \in C_i^\mu$ and (ii) $p_i < p_j$ for all $j \in S_i$ such that $jP_i\mu(i)$. Choose any $i \in N$. Since i and $\mu(i)$ belong to the same trading cycle at μ, we have $p_i = p_{\mu(i)}$. Since $\mu(i) \in S_i$, $\mu(i)$ is in agent i's budget set. Moreover, since $p_j > p_i$ for all $j \in S_i$ such that $jP_i\mu(i)$, $\mu(i)$ is optimal for agent i among assignments in his budget set. Hence, p supports μ. $\square$

In order for a price vector to support a given matching, the prices of all agents in a trading cycle should be equal. This is because each agent's endowment should be in the budget set of the preceding agent along the cycle, which is possible only when all the agents in the cycle have the same price. Moreover, all assignments that are preferred to an agent's assignment at the matching should be outside of the agent's budget set. These two properties are not only necessary but also sufficient for a price vector to support a given matching. Note that it does not matter whether those that are less preferred to an agent's assignment belong to the agent's budget set or not. In the following proposition, we study the relationship between competitive matchings and the core.

[4] Since a matching can be considered as a permutation on a finite set, this result follows from the cycle decomposition theorem for permutations (see, for example, Hungerford, 1974, Theorem 6.3, Ch. I).

Proposition 1. *Every competitive matching is in the core.*

Proof. Suppose that $\mu \in \mathscr{M}^f$ is a competitive matching with supporting price vector p. Suppose to the contrary that μ is not in the core. Then there exist a coalition $T \subseteq N$ and a matching $\mu' \in \mathscr{M}^f$ such that μ' dominates μ via T. Since $\mu'(i) R_i \mu(i)$ and $\mu'(i) \in S_i$ for all $i \in T$, we have $p_{\mu'(i)} \geq p_{\mu(i)} = p_i$ for all $i \in T$ by Lemma 1. Also, since $\mu'(i) P_i \mu(i)$ for some $i \in T$, we have $p_{\mu'(i)} > p_{\mu(i)} = p_i$ for some $i \in T$. Summing these inequalities over $i \in T$, we obtain $\sum_{i \in T} p_{\mu'(i)} > \sum_{i \in T} p_i$. However, since $\mu'(i) \in T$ for all $i \in T$ and μ' is a bijection, we should have $\sum_{i \in T} p_{\mu'(i)} = \sum_{i \in T} p_i$, which is a contradiction. □

Proposition 1 shows that the set of competitive matchings is a subset of the core. Thus, matchings in the core are candidates for competitive matchings, and if the core is empty, there is no competitive matching. In the following theorem, we study further the relationship between competitive matchings and the core assuming that a competitive matching exists.

Theorem 1. *If there exists a competitive matching, then the core is a singleton consisting of the competitive matching.*

Proof. Suppose that $\mu \in \mathscr{M}^f$ is a competitive matching with supporting price vector p. Let K be the number of distinct prices in p. Note that K is finite with $K \leq |N|$. We can partition N as $\{N^1, \ldots, N^K\}$, where N^k is the set of agents whose prices are the kth highest for all $k = 1, \ldots, K$. By Lemma 1, N^k can be partitioned into one or more trading cycles at μ for all $k = 1, \ldots, K$.

By Proposition 1, μ is in the core. Choose any matching $\mu' \in \mathscr{M}^f$ in the core. Choose any trading cycle $C \subseteq N^1$ at μ and any $j \in C$. Suppose that $\mu'(j) \neq \mu(j)$. Since agents in N^1 can afford any assignment, $\mu(i) R_i i'$ for all $i' \in S_i$, for all $i \in C$. Hence, $\mu(i) R_i \mu'(i)$ for all $i \in C$ and $\mu(j) P_j \mu'(j)$. Since C is a trading cycle at μ, we have $\mu(i) \in C$ for all $i \in C$. Hence, μ dominates μ' via C, a contradiction. It follows that $\mu' = \mu$ on C, and thus on N^1.

Now choose any trading cycle $C \subseteq N^2$ at μ and any $j \in C$. Suppose that $\mu'(j) \neq \mu(j)$. Since agents in N^2 can afford any assignment other than those in N^1, $\mu(i) R_i i'$ for all $i' \in S_i \setminus N^1$, for all $i \in C$. Also, since $\mu' = \mu$ on N^1, $\mu'(i) \notin N^1$ for all $i \in C$. Hence, $\mu(i) R_i \mu'(i)$ for all $i \in C$ and $\mu(j) P_j \mu'(j)$. Again, μ dominates μ' via C, a contradiction. It follows that $\mu' = \mu$ on C, and thus on N^2.

Repeating this argument, we can show that $\mu' = \mu$ on N^k, for all $k = 1, \ldots, K$. Thus, $\mu' = \mu$, and there cannot be any other matching than μ in the core. □

Combined with Proposition 1, Theorem 1 shows that, if the set of competitive matchings is nonempty, it coincides with the core and contains exactly one matching. Thus, there can be at most one competitive matching. Theorem 1 also shows that a singleton core is necessary for the existence of competitive equilibria. However, as shown in Example 2 below, a singleton core is not sufficient for the existence of competitive equilibria. Below we present two examples of marriage problems to illustrate that a marriage problem may or may not have a competitive matching.

Example 1. Consider a marriage problem with $M = \{1, 2\}$, $W = \{3, 4\}$, and preferences $P_1 : 3, 4, 1$, $P_2 : 3, 4, 2$, $P_3 : 1, 2, 3$, $P_4 : 2, 1, 4$. It can be checked that the marriage problem has a unique stable matching $\mu = \{(1, 3), (2, 4)\}$.[5] At the stable matching, every agent except for agent 2 obtains his or her favorite partner. Hence, any price vector p satisfying $p_1 = p_3 > p_2 = p_4$ supports μ, and thus μ is a competitive matching.

Example 2. Consider a marriage problem with $M = \{1, 2\}$, $W = \{3, 4\}$, and preferences $P_1 : 4, 3, 1$, $P_2 : 3, 4, 2$, $P_3 : 1, 3, 2$, $P_4 : 2, 4, 1$. In this marriage problem, each woman has only one acceptable man. Thus, the stable matching is unique and assigns each woman to her acceptable man, i.e., $\mu = \{(1, 3), (2, 4)\}$. The stable matching μ is not competitive because there is no price vector supporting μ. To see why, suppose that there is one, say p. By Lemma 1, we have $p_1 = p_3$ and $p_2 = p_4$. Also, $4P_1[3 = \mu(1)]$ and $3P_2[4 = \mu(2)]$ imply $p_4 > p_1$ and $p_3 > p_2$, respectively. Combining these equalities and inequalities yields a contradiction.

Two questions arise from Theorem 1: When does a competitive matching exist? If a competitive matching exists, how can we find the unique competitive matching? Our next result (Theorem 2) addresses these questions. In answering these questions, the *top trading cycles algorithm* developed by David Gale and introduced in Shapley and Scarf (1974) plays an important role. Although the top trading cycles algorithm is originally developed for housing markets, it can be applied to generalized matching problems without modification, as described below.[6]

- *Step 1.* Each agent points to the owner of the good he prefers most. Since there are a finite number of agents, there exists at least one cycle of agents pointing to one another. Each agent in a cycle is assigned the endowment of the agent he points to and exits from the market. If there is at least one remaining agent, proceed to the next step. Otherwise, stop.

  In general, at:

- *Step l.* Each remaining agent points to the owner of the good he prefers most among the remaining goods. Each agent in a cycle is assigned the endowment of the agent he points to and exits from the market. If there is at least one remaining agent, proceed to the next step. Otherwise, stop.

[5] Recall that a matching can be represented as the collection of trading cycles at the matching.

[6] See Moulin (1995, Sec. 3.2) and Abdulkadiroğlu and Sönmez (2013, Sec. 3.1) for a description of the top trading cycles algorithm and its properties for housing markets.

Consider an agent who remains in the market at a step. Since he has strict preferences and has an option to choose his own endowment, there exists a unique good he prefers most among those available at that step. Hence, the choice of each remaining agent at any step is determined uniquely. Since each remaining agent points to only one agent, each agent belongs to at most one cycle. There is at least one cycle at each step, while there can be multiple cycles formed at a step. In case of multiple cycles, the algorithm eliminates agents in all of them simultaneously. Thus, at each step exiting agents are existent and determined uniquely with their assignments. Also, since there are a finite number of agents, the algorithm must terminate after at most $|N|$ steps. Combining agents' assignments at the end, we obtain a unique matching $\mu \in \mathscr{M}$ produced by the algorithm.

Let T^l be the set of agents who exit at Step l of the top trading cycles algorithm, for all $l = 1, \ldots, L$, where L denotes the total number of steps. Then the above argument shows that the algorithm also produces a unique partition $\{T^1, \ldots, T^L\}$ of N. Given a matching problem $G = (N, S, R)$, a subset $\tilde{N} \subseteq N$ of agents defines a *subproblem* where agent i' preferences R_i are restricted to $S_i \cap \tilde{N}$ for all $i \in \tilde{N}$. We say that a sequence $(i_0, i_1, \ldots, i_{K-1})$ or a set $\{i_0, i_1, \ldots, i_{K-1}\}$ is a *top trading cycle* for the subproblem $\tilde{N}$ if i_{k+1} is i_k's most preferred assignment among those in $\tilde{N}$ for all $k = 0, \ldots, K-1$ where $i_K = i_0$. By the construction of the algorithm, T^l consists of one or more top trading cycles for the subproblem $N \setminus (\cup_{k \leq l-1} T^k)$, for all $l = 1, \ldots, L$.

Let us illustrate how the top trading cycles algorithm works with the two previous examples. First, consider the marriage problem in Example 1. At Step 1, both agents 1 and 2 point to agent 3, agent 3 points to agent 1, and agent 4 points to agent 2. Here we obtain a top trading cycle $(1, 3)$ for $N = M \cup W$, and agents 1 and 3 are matched and exit (i.e., $T^1 = \{1, 3\}$). At Step 2, the remaining agents, agents 2 and 4, point to each other. So they form a top trading cycle $(2, 4)$ for $N \backslash T^1$, are matched, and exit (i.e., $T^2 = \{2, 4\}$). Since there is no remaining agent, the algorithm terminates after Step 2, yielding the matching $\mu = \{(1, 3), (2, 4)\}$. Next, consider the marriage problem in Example 2. At Step 1, agent 1 points to agent 4, agent 2 points to agent 3, agent 3 points to agent 1, and agent 4 points to agent 2. We obtain a top trading cycle $(1, 4, 2, 3)$ for $N = M \cup W$, and all the agents exit (i.e., $T^1 = \{1, 2, 3, 4\}$). The algorithm terminates after Step 1, yielding the matching $\mu = \{(1, 4, 2, 3)\}$.

The matching obtained from the top trading cycles algorithm does not need to be feasible, and whether it is feasible or not depends on the specification of $\mathscr{M}^f$. For instance, the resulting matching in Example 1 is bilateral and thus feasible, whereas it is not the case in Example 2. The following theorem shows that a competitive equilibrium exists exactly when the matching produced by the top trading cycles algorithm is feasible. This result is consistent with our previous observation that the marriage problem in Example 1 has a competitive matching while that in Example 2 does not.

Theorem 2. *There exists a competitive matching if and only if the matching μ obtained from the top trading cycles algorithm is feasible (i.e., $\mu \in \mathscr{M}^f$), in which case μ is the unique competitive matching.*

Proof. ($\Leftarrow$) Suppose that the top trading cycles algorithm yields a feasible matching $\mu \in \mathscr{M}^f$. Let $\{T^1, \ldots, T^L\}$ be the partition of N that we obtain from the top trading cycles algorithm. Define a price vector $p \in \mathbb{R}^N_+$ by $p_i = L - l$ if $i \in T^l$. Choose any $i \in N$. Since the trading cycle C_i^μ at μ is formed as a top trading cycle at some step of the algorithm, all the agents in C_i^μ exit at the same step, and we have $p_i = p_j$ for all $j \in C_i^\mu$. Choose any $j \in S_i$ such that $j P_i \mu(i)$. Since agent i points to $\mu(i)$ at the step when he exits, agent j should not be available at that step. Thus, $p_j > p_i$. By Lemma 1, (p, μ) is a competitive equilibrium, and μ is the unique competitive matching.

($\Rightarrow$) Suppose that there exists a (unique) competitive matching $\mu \in \mathscr{M}^f$ with supporting price vector p. Let $\{C^1, \ldots, C^K\}$ be the partition of N consisting of trading cycles at μ. By Lemma 1, agents in a trading cycle have the same price. Let q^k be the common price of agents in trading cycle C^k, for all $k = 1, \ldots, K$. Without loss of generality, we assume that $q^1 \geq q^2 \geq \cdots \geq q^K$. For each agent $i \in C^1$, $\mu(i)$ is agent i's most preferred assignment among those in N. Hence, C^1 is a top trading cycle for N, and each agent $i \in C^1$ obtains $\mu(i)$ and exits from the market at Step 1 of the top trading cycles algorithm. For each agent $i \in C^2$, $\mu(i)$ is agent i's most preferred assignment among those in $N \setminus C^1$. If no agent $i \in C^2$ prefers an assignment in C^1 to $\mu(i)$, C^2 is also a top trading cycle for N and agents in C^2 exit at Step 1. Otherwise, C^2 is a top trading cycle for $N \setminus C^1$ and agents in C^2 exit at Step 2. In general, suppose that we have proceeded with the top trading cycles algorithm to eliminate trading cycles up to C^{k-1}. Let l be the number of the step at which the trading cycle C^{k-1} is eliminated. Then we have the sets $T^1, \ldots, T^l$ where T^j is the set of agents who exit at Step j, for $j = 1, \ldots, l$. Consider the trading cycle C^k. For each agent $i \in C^k$, $\mu(i)$ is agent i's most preferred assignment among those in $N \setminus (\cup_{j \leq k-1} C^j)$. If no agent $i \in C^k$ prefers an assignment in T^l to $\mu(i)$, C^k is a top trading cycle for $N \setminus (\cup_{j \leq l-1} T^j)$ and agents in C^k exit at Step l, in which case we add agents in C^k to T^l. Otherwise, C^k is a top trading cycle for $N \setminus (\cup_{j \leq l} T^j)$ and agents in C^k exit at Step $l+1$, in which case we set $T^{l+1} = C^k$. As we proceed in this way, we can see that each T^j is the union of one or more trading cycles at μ with consecutive indexes, while for every $j' < j$, $T^{j'}$ contains trading cycles with lower indexes than those in T^j. Also, each agent i obtains assignment $\mu(i)$ when he exits, and thus the top trading cycles algorithm yields the matching μ. □

A competitive equilibrium exists exactly when the top trading cycles algorithm yields a feasible matching. If a competitive equilibrium exists, we can find the unique competitive matching using the top trading cycles algorithm. We can also find a supporting price vector by assigning a higher price to agents in a top trading cycle formed at an earlier step of the algorithm. The same construction of a supporting price vector can be found in Shapley and Scarf (1974, Sec. 6). This construction guarantees that each agent's assignment is the most preferred among affordable choices. It is straightforward to see that the order of prices in a supporting price vector is uniquely determined (in other words, a supporting price vector is unique in the ordinal sense) if and only if there is exactly one top trading cycle at each step of the top trading cycles algorithm.

Theorem 2 generalizes the existing result on housing markets that the unique competitive matching can be obtained by applying the top trading cycles algorithm (Roth and Postlewaite, 1977). The difference is that a competitive equilibrium always exists in housing markets while a competitive equilibrium may not exist in generalized matching problems. The matching produced by the top trading cycles algorithm can be considered as a Pareto-efficient allocation when there is no restriction on possible trades. Hence, Theorem 2 can be interpreted as follows: A competitive equilibrium exists when there is no efficiency loss due to restrictions on trades imposed by $\mathscr{M}^f$. This suggests that price mechanisms work better when there are less restrictions on possible trades. We can see from Theorem 2 that, as the set $\mathscr{M}^f$ of feasible matchings becomes more restrictive, the chance of having a competitive equilibrium gets smaller. In contrast, as we expand $\mathscr{M}^f$, it becomes more likely that a competitive equilibrium exists. In the extreme case where every matching is feasible, i.e, $\mathscr{M}^f = \mathscr{M}$, we always have a competitive equilibrium.

Corollary 1. *If $\mathscr{M}^f = \mathscr{M}$, then the core is a singleton and the matching in the core is the unique competitive matching.*

Corollary 1 covers housing markets, which have $S_i = N$ for all $i \in N$, while it holds regardless of S. Thus, the crucial feature of housing markets that guarantees the existence of competitive equilibria is that there is no restriction on feasible trades.[7]

Now we study the implications of Theorem 2 for bilateral matching problems, which include roommate and marriage problems. Note that every trading cycle at a bilateral matching has length 1 or 2.[8] Thus, Theorem 2 can be rewritten as follows in the context of bilateral matching problems.

[7]There is a variant of the top trading cycles algorithm adapted for school choice problems where students have preferences over schools and schools have capacities and priorities over students. Roughly speaking, the top trading cycles algorithm for school choice problems allows students who have the highest priority at some school to trade their priorities with each other (see Abdulkadiroğlu and Sönmez, 2003, for a description of the algorithm and its properties). As in housing markets, there is no restriction on feasible trades of priorities, and thus the algorithm always yields a feasible matching.

[8]We define the length of a cycle as the number of distinct elements in the cycle.

Corollary 2. *Suppose that $\mathscr{M}^f = \mathscr{M}^b$. Then a competitive equilibrium exists if and only if every top trading cycle obtained at each step of the top trading cycles algorithm has length 1 or 2.*

If we obtain a top trading cycle of length 3 or more at some step of the top trading cycles algorithm, as in Example 2, the bilateral matching problem does not have a competitive equilibrium. In other words, a bilateral matching problem has a competitive equilibrium when bilateral trades are sufficient to realize all gains from trade. For marriage problems, Theorem 2 also implies the following: a competitive equilibrium exists if and only if the top trading cycles algorithm yields the same matching as (either version of) the deferred acceptance algorithm.[9]

4 Comparison of Conditions in Bilateral Matching Problems

4.1 Roommate Problems

Our analysis in Section 3 shows that, if a competitive matching exists in a roommate problem, then it is the unique stable matching. Hence, we can regard the existence of competitive matchings as a sufficient condition for the existence and uniqueness of stable matchings. In the literature, several studies have proposed various conditions on agents' preferences for the existence and uniqueness of stable matchings in roommate problems. In this subsection, we study the relationships between our condition based on competitive equilibria and some existing conditions.

We first introduce some concepts that are used to state various conditions. We refer to top trading cycles of length 1 and 2 as *top trading singles and pairs*, respectively.[10] A *ring* in a matching problem is an ordered list of agents $(i_1, i_2, \ldots, i_K)$ with $K \geq 3$ such that $i_{k+1} P_{i_k} i_{k-1} P_{i_k} i_k$ (subscripts modulo K) for all $k = 1, \ldots, K$. A *cycle* in a matching problem is an ordered list of agents $(i_1, i_2, \ldots, i_K)$ with $K \geq 3$ such that $i_{k+1} P_{i_k} i_{k-1}$ (subscripts modulo K) for all $k = 1, \ldots, K$. Below we present several conditions on agents' preferences.

Definition 1. (i) A matching problem is *α–reducible* if there is a top trading single or pair for every subproblem.[11]

[9] A related, though different, result can be found in Kesten (2006), who shows that, in priority-based allocation problems, the top trading cycles algorithm (adapted to the context as for school choice) and the (agent-proposing) deferred acceptance algorithm yield the same matching if and only if the priority structure is acyclic.

[10] Alternative names for top trading pairs in the literature include "pairs of P-reciprocal agents" in Alcalde (1995), "fixed pairs" in Clark (2006), and "top-top matches" in Niederle and Yariv (2009).

[11] Alcalde's (1995) definition of α–reducibility considers only top trading pairs but not singles because he assumes that there are an even number of agents and that every agent is acceptable to all the others. Niederle and Yariv (2009) use the *top-top match property* instead of α–reducibility for the same meaning in

(ii) A matching problem is *iteratively α–reducible* if every top trading cycle obtained at each step of the top trading cycles algorithm is a top trading single or pair.

(iii) A matching problem satisfies the *no odd rings condition* if there is no ring $(i_1, i_2, \ldots, i_K)$ such that K is odd.

(iv) A matching problem has *acyclic preferences* if there is no cycle.

(v) A matching problem satisfies the *symmetric utilities hypothesis* if there exists a symmetric function $u : N^2 \to \mathbb{R}$ representing agents' preferences, that is, $u(i,j) = u(j,i)$ for all $i, j \in N$ and $u(i,j) > u(i,j')$ if and only if jP_ij' for all $i, j, j' \in N$.

Note that the properties in Definition 1 apply to any matching problem $G = (N, S, R)$ regardless of the set $\mathscr{M}^f$ of feasible matchings. Below we summarize the implications of the above properties for roommate problems.[12] Tan (1991) shows that there exists a stable matching if and only if there is a "stable partition" without odd rings.[13] Chung (2000) shows that the no odd rings condition is sufficient for the existence of stable matchings. Any ring in a marriage problem must have an even length, and thus the no odd rings condition is satisfied for any marriage problem, which explains the existence result established by Gale and Shapley (1962).[14] Rodrigues-Neto (2007) proves that a matching problem satisfies the symmetric utilities hypothesis if and only if it has acyclic preferences, in which case there is a unique stable matching. Alcalde (1995) shows that α–reducibility is sufficient for the uniqueness of stable matchings. The idea behind the uniqueness result is that, if there is a pair of agents who prefer each other most, they must be matched under any stable matching, and we can apply this argument iteratively. By the same logic, we can show that iterative α–reducibility implies the uniqueness of stable matchings, which is also confirmed by Theorem 1 and Corollary 2. In the following proposition, we compare the conditions in Definition 1.

Proposition 2. *(i) If a matching problem is α-reducible, then it is iteratively α–reducible. If a matching problem has acyclic preferences, then it is α–reducible. If a matching problem has acyclic preferences, then it satisfies the no odd rings condition. The converses of the three implications do not hold.*

(ii) The no odd rings condition is neither stronger nor weaker than (iterative) α–reducibility, while there is a matching problem that satisfies both conditions.

the context of marriage problems. α–reducibility corresponds to the top-coalition property in Banerjee et al. (2001) who consider a more general model than bilateral matching problems. Note that, in bilateral matching problems where agents form coalitions of size one or two, there is no difference between the top-coalition property and the weak top-coalition property of Banerjee et al. (2001).

[12]See Gudmundsson (2014) for an excellent review of various conditions for the existence of stable matchings in roommate problems allowing for weak preferences.

[13]See Tan (1991) for the definition of stable partitions, which generalize stable matchings.

[14]As pointed out by Chung (2000, Definition 3), any marriage problem can be expressed as a roommate problem by putting all the other agents on the same side at the bottom of each agent's preference list.

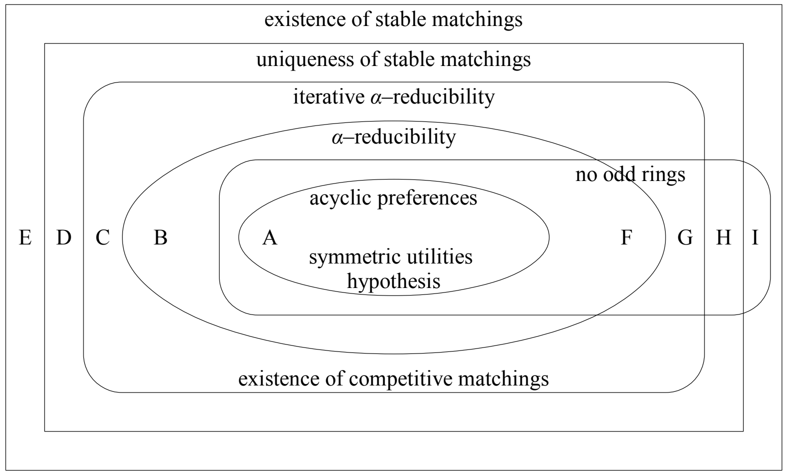


Figure 1: Conditions for the existence and uniqueness of stable matchings.

Proof. (i) It is clear from the definitions that α–reducibility implies iterative α–reducibility. Suppose that the symmetric utilities hypothesis is satisfied with function u. For any subproblem $\tilde{N}$, let (i^*, j^*) be a solution to $\max_{(i,j)\in\tilde{N}^2} u(i,j)$. Then $\{i^*, j^*\}$ is a top trading single or pair for the subproblem $\tilde{N}$. Since the symmetric utilities hypothesis is equivalent to acyclic preferences (Rodrigues-Neto, 2007), acyclic preferences imply α–reducibility. Since a ring is a cycle, acyclic preferences imply the no odd rings condition. We can prove that the converses do not hold by showing that there are matching problems belonging to areas C, B, and F in Figure 1. This is done in the discussion on Examples 4, 3, and 7 below.

(ii) This result is proven once we show that there are matching problems belonging to areas H, B, and A in Figure 1. This is done in the discussion on Examples 2, 3, and 1 below. □

Proposition 2 shows, among others, that iterative α–reducibility is a weaker sufficient condition for uniqueness than existing ones such as α–reducibility and acyclic preferences.[15] The relationships between various conditions, including those shown in Proposition 2, are depicted in Figure 1.

We show by example that there exists a matching problem belonging to each area in Figure 1. First, consider the matching problem induced by the marriage problem in Example 1. It can be checked that the matching problem has acyclic preferences, and thus it belongs to area A. This shows that the set of matching problems with acyclic preferences is nonempty and that the intersection of any two regions in Figure 1 is nonempty.

[15] The kind of weakening from α–reducibility to iterative α–reducibility is also suggested in footnote 11 of Banerjee et al. (2001) and in Gudmundsson (2014).

Example 3 (from Gudmundsson, 2014)**.** Consider $N = \{1, 2, 3, 4, 5\}$ and preferences P_1 : $3, 2, 5, 4, 1$, $P_2 : 4, 3, 1, 5, 2$, $P_3 : 1, 4, 2, 5, 3$, $P_4 : 2, 5, 3, 1, 4$, $P_5 : 2, 1, 4, 3, 5$. There is an odd ring, $(1, 2, 3, 4, 5)$, which is also a cycle, and it can be verified that the matching problem is α–reducible. Hence, the problem belongs to area B. This shows that α–reducibility is strictly weaker than acyclic preferences and not stronger than the no odd rings condition.

Example 4 (from Example 2 of Chung, 2000)**.** Consider $N = \{1, 2, 3, 4\}$ and preferences $P_1 : 2, 1, 4, 3$, $P_2 : 1, 3, 4, 2$, $P_3 : 4, 2, 3, 1$, $P_4 : 2, 3, 4, 1$. There is an odd ring, $(2, 3, 4)$. The matching problem is iteratively α–reducible with $T^1 = \{1, 2\}$ and $T^2 = \{3, 4\}$. However, it is not α–reducible because there is neither a top trading single nor a top trading pair for the subproblem $\tilde{N} = \{2, 3, 4\}$. Hence, the problem belongs to area C. This shows that iterative α–reducibility is strictly weaker than α–reducibility and not stronger than the no odd rings condition.

Example 5. Consider the matching problem in Example 4 with agent 1's preferences modified as $P_1 : 4, 2, 1, 3$. Then $(2, 3, 4)$ is still an odd ring, and the matching $\mu = \{(1, 2), (3, 4)\}$ is the unique stable matching. However, there is neither a top trading single nor a top trading pair for the matching problem, and thus it is not iteratively α–reducible. Hence, the problem belongs to area D. This shows that the uniqueness of stable matchings does not imply iterative α–reducibility.

Example 6. Consider $N = \{1, 2, 3, 4\}$ and preferences $P_1 : 3, 2, 4, 1$, $P_2 : 1, 3, 4, 2$, P_3 : $4, 1, 2, 3$, $P_4 : 2, 3, 1, 4$. There is an odd ring, $(2, 3, 4)$. There are two stable matchings $\mu = \{(1, 2), (3, 4)\}$ and $\mu' = \{(1, 3), (2, 4)\}$. Hence, the problem belongs to area E. This shows that the existence of stable matchings does not imply uniqueness.

Example 7. Consider $N = \{1, 2, 3\}$ and preferences $P_1 : 1, 2, 3$, $P_2 : 2, 3, 1$, $P_3 : 3, 1, 2$. There is no odd ring, while there is a cycle $(1, 2, 3)$. Also, the matching problem is α–reducible. Hence, the problem belongs to area F. This shows that the no odd rings condition does not imply acyclic preferences.

Example 8. Consider $N = \{1, 2, 3, 4\}$ and preferences $P_1 : 2, 1, 4, 3$, $P_2 : 3, 1, 2, 4$, P_3 : $4, 1, 3, 2$, $P_4 : 3, 4, 2, 1$. There is no odd ring. The matching problem is iteratively α–reducible with $T^1 = \{3, 4\}$ and $T^2 = \{1, 2\}$. However, it is not α–reducible because there is neither a top trading single nor a top trading pair for the subproblem $\tilde{N} = \{1, 2, 3\}$. Hence, the problem belongs to area G. This shows that the no odd rings condition does not imply α–reducibility.

Consider the matching problem induced by the marriage problem in Example 2. We have already seen that the matching problem has a unique stable matching but no competitive matching. Since the matching problem is a marriage problem, it satisfies the no odd rings condition. Hence, it belongs to area H. This shows that the no odd rings condition does not imply iterative α–reducibility.

Example 9. Consider $N = \{1, 2, 3, 4\}$ and preferences $P_1 : 2, 4, 3, 1,\ P_2 : 3, 1, 4, 2,\ P_3 : 4, 2, 1, 3,\ P_4 : 1, 3, 2, 4$. There is no odd ring. There are two stable matchings $\mu = \{(1, 2), (3, 4)\}$ and $\mu' = \{(1, 4), (2, 3)\}$. Hence, the problem belongs to area I. This shows that the no odd rings condition does not imply the uniqueness of stable matchings.

Examples 2 and 7–9 also show that the strict inclusions shown in Examples 3–6 continue to hold even when we restrict attention to the domain of the no odd rings condition.

4.2 Marriage Problems

In this subsection, we focus on marriage problems. Since all marriage problems satisfy the no odd rings condition, we work within the region of the no odd rings condition in Figure 1, and there exists a stable matching. In the literature, several studies have provided sufficient conditions for the uniqueness of stable matchings in marriage problems. We first review some of the existing conditions.

Definition 2. (i) A marriage problem satisfies the *sequential preference condition (SPC)* if for each $m_i \in M$, $w_i P_{m_i} w_j$ for all $j > i$, and for each $w_i \in W$, $m_i P_{w_i} m_j$ for all $j > i$.

(ii) A marriage problem satisfies the *no crossing condition (NCC)* if for any $i < j$ and $k < l$, $w_j P_{m_k} w_i$ implies $w_j P_{m_l} w_i$, and $m_l P_{w_i} m_k$ implies $m_l P_{w_j} m_k$.

(iii) A marriage problem has *aligned preferences* if there exists an ordinal potential $\widetilde{P}$ defined on $M \times W$ such that $w P_m w'$ implies $(m, w)\widetilde{P}(m, w')$ and $m P_w m'$ implies $(m, w)\widetilde{P}(m', w)$.

(iv) A marriage problem has *no simultaneous cycles* if there are no ordered lists of men and women $(m_1, m_2, \ldots, m_K)$ and $(w_1, w_2, \ldots, w_K)$ with $K \geq 2$ such that $w_k P_{m_k} w_{k-1} P_{m_k} m_k$ and $m_k P_{w_k} m_{k-1} P_{w_k} w_k$ (subscripts modulo K) for all $k = 1, \ldots, K$.

The SPC is satisfied if men and women can be ordered so that each man (resp. woman) prefers the woman (resp. man) with the same rank as his (resp. hers) to those with a lower rank. Eeckhout (2000) considers marriage problems where there are equal numbers of men and women and all men are acceptable to all women and vice versa, and he shows that, under the SPC, the matching that pairs the man and woman with the same rank is the unique stable matching. The NCC is satisfied if men and women can be ordered so that if a man (resp. woman) prefers a woman (resp. man) to another one with a higher rank than hers (resp. his), so does a man (resp. woman) with a lower rank than his (resp. hers). Its name comes from the following property: When we position any two men and two women along two lines according to their orderings and draw arrows representing the men and women's preferred partners, these arrows cannot cross each other under the NCC. Clark (2006) considers the same setup as in Eeckhout (2000) and shows that the NCC is sufficient for the uniqueness of stable matchings. Niederle and Yariv (2009) show that there exists a unique stable matching if the marriage problem has aligned preferences. A simultaneous cycle in a marriage problem corresponds to a ring. Recently, Romero-Medina and Triossi (2013) show that the absence of simultaneous cycles implies the uniqueness of stable matchings.[16] Note that the conditions in Definition 2 are defined in the context of marriage problems while those in Definition 1 apply to any matching problems. In the following proposition, we compare the conditions in Definitions 1 and 2.

Proposition 3. *Consider marriage problems where there are equal numbers of men and women and all men are acceptable to all women and vice versa.*

(i) A marriage problem satisfies the SPC if and only if it is iteratively α–reducible.

(ii) A marriage problem has aligned preferences if and only if it has acyclic preferences. A marriage problem has no simultaneous cycles if and only if it has acyclic preferences.

(iii) If a marriage problem satisfies the NCC, then it is α–reducible. The converse does not hold. The NCC is neither stronger nor weaker than aligned preferences, while there is a marriage problem that satisfies both conditions.

Now consider marriage problems in general.

(iv) The condition of no simultaneous cycles is neither stronger nor weaker than (iterative) α–reducibility, while there is a matching problem that satisfies both conditions.

Proof. (i) Suppose that the marriage problem is iteratively α–reducible. Then at each step of the top trading cycles algorithm, there exists at least one top trading pair. We rank these pairs in the order they exit. If there are multiple pairs exiting at the same step, we assign their ranks arbitrarily. Then by construction, each man (resp. woman) prefers the woman (resp. man) with the same rank to any woman (resp. man) with a lower rank. Hence, the SPC is satisfied. Suppose that the SPC holds. Then when we apply the top trading cycles algorithm, the pairs (m_i, w_i) exit in order of their ranks, although some pairs may exit simultaneously. Thus, the marriage problem is iteratively α–reducible.

(ii) The first statement follows from the characterization of ordinal potential games in Voorneveld and Norde (1997), as mentioned in Niederle and Yariv (2009). Since all men are acceptable to all women and vice versa, rings coincide with cycles. Thus, the condition of no simultaneous cycles is equivalent to acyclic preferences.

(iii) The result that the NCC implies α–reducibility is shown in Clark (2006, Theorem 3). Consider a marriage problem where all men have the same preferences over women

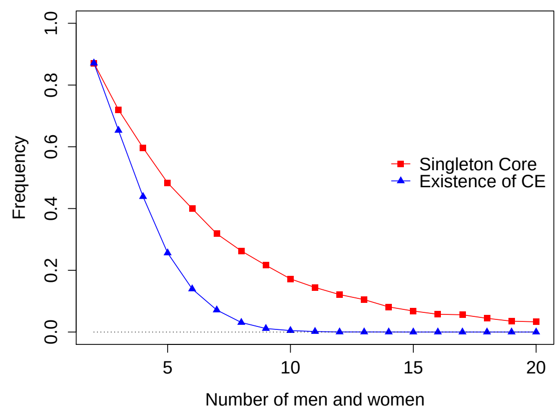
[16] Niederle and Yariv (2009) assume that all men are acceptable to all women and vice versa while allowing unequal numbers of men and women. Romero-Medina and Triossi (2013) allow not only unequal numbers of men and women but also unacceptable agents.

and vice versa. Then it can be checked that the marriage problem satisfies the NCC and has aligned preferences. Next consider the marriage problem with $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, and preferences $P_{m_i} : w_1, w_2, w_3, m_i$ for $i = 1, 2, 3$, $P_{w_1} : m_1, m_2, m_3, w_1$, $P_{w_2} : m_2, m_3, m_1, w_2$, $P_{w_3} : m_3, m_1, m_2, w_3$. Since all men have the same preferences over women, the marriage problem has aligned preferences, as shown by Niederle and Yariv (2009). However, for any orderings of men and women, we can find pairs of men and women such that a crossing occurs, and thus the NCC does not hold. This example shows that the NCC is not weaker than aligned preferences. Since acyclic preferences, which are equivalent to aligned preferences, imply α–reducibility, it also shows that the converse of the first statement does not hold. Lastly, consider the marriage problem with $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, and preferences $P_{m_1} : w_1, w_2, w_3, m_1$, $P_{m_i} : w_2, w_3, w_1, m_i$ for $i = 2, 3$, $P_{w_i} : m_2, m_1, m_3, w_i$ for $i = 1, 2$, $P_{w_3} : m_3, m_2, m_1, w_3$. The marriage problem has a cycle, $(w_2, m_1, w_1, m_2, w_3, m_3)$, and thus it does not have aligned preferences. However, it can be checked that the NCC is satisfied with the given orderings of men and women. This example shows that the NCC is not stronger than aligned preferences.

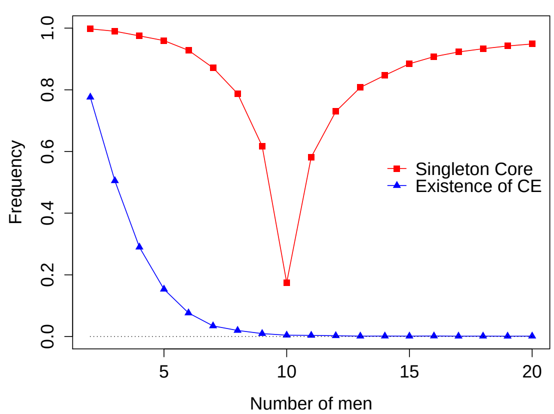
(iv) We have seen that the marriage problem in Example 1 has acyclic preferences. Thus, it has no simultaneous cycles and satisfies (iterative) α–reducibility. We have also seen that the marriage problem in Example 2 is not iteratively α–reducible. Since each woman has only one acceptable man, there is no ring. Thus, the marriage problem has no simultaneous cycles, which shows that the condition of no simultaneous cycles is not stronger than (iterative) α–reducibility. Lastly, consider the marriage problem with $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, and preferences $P_{m_1} : w_1, w_3, w_2, m_1$, $P_{m_2} : w_2, w_1, w_3, m_2$, $P_{m_3} : w_3, w_2, w_1, m_3$, $P_{w_1} : m_1, m_3, m_2, w_1$, $P_{w_2} : m_2, m_1, m_3, w_2$, $P_{w_3} : m_3, m_2, m_1, w_3$. Then the marriage problem has a ring, $(m_1, w_3, m_2, w_1, m_3, w_2)$, and it can be checked that it is α–reducible. Thus, this example shows that the condition of no simultaneous cycles is not weaker than (iterative) α–reducibility. □

Proposition 3 shows that, in the class of marriage problems considered in Eeckhout (2000), the SPC is equivalent to iterative α–reducibility, while the NCC, aligned preferences, and the absence of simultaneous cycles are stronger than α–reducibility. Hence, the existence of competitive equilibria can be considered as one of weaker sufficient conditions for the uniqueness of stable matchings in marriage problems. Proposition 3 also shows that, in general marriage problems, the absence of simultaneous cycles is different from (iterative) α–reducibility, as mentioned in Romero-Medina and Triossi (2013).

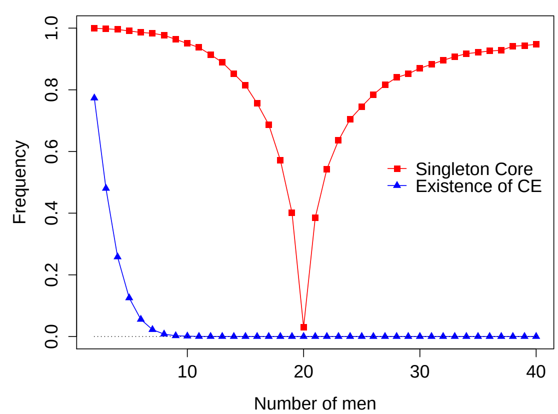
Since singleton cores are only necessary but not sufficient for the existence of competitive matchings, it is of interest to examine how restrictive the latter condition is relative to the former. In Figure 2, we present numerical results to address this issue in the context of marriage problems. We consider marriage problems in which all men are acceptable to all



(a)



(b)



(c)

Figure 2: Frequencies of having a singleton core and a competitive matching in marriage problems with randomly generated preferences. In (a), the numbers of men and women are equal and varied from 2 to 20. In (b), the number of men is varied from 2 to 20 while the number of women is fixed at 10. In (c), the number of men is varied from 2 to 40 while the number of women is fixed at 20. For each combination of the numbers of men and women, 10,000 marriage problems are generated.

women and vice versa and agents' preferences are generated independently and uniformly at random, as in Ashlagi et al. (2015). In Figure 2(a), we study balanced marriage problems where there are equal numbers of men and women. As the numbers of men and women increase, the frequencies of having a singleton core and a competitive matching both reduce but the latter exhibits a faster decline. In Figure 2(b) and (c), we consider unbalanced marriage problems where there are unequal numbers of men and women. As marriage problems become more unbalanced, the frequency of having a singleton core increases and approaches 1. In contrast, as the number of men increases with the number of women held fixed, the frequency of having a competitive matching decreases to 0.

Overall, we can see that as there are more agents on one or both sides, it becomes less likely that a competitive matching exists and that with 10 or more agents on each side, it is very unlikely to have a competitive matching. The main reason for nonexistence can be explained as follows. As there are more agents, they have more potential partners. With random preferences, it becomes less likely to have two agents who prefer each other most, and top trading cycles tend to have a length longer than two. In other words, with many agents (and thus many goods), all gains from trade can be exploited when we allow trades involving more than two agents, and as a result, a competitive equilibrium does not exist when we restrict trades to be bilateral. The observations from the numerical results suggest that, at least in marriage problems with random preferences, centralized matching mechanisms will work more successfully than decentralized price mechanisms.

5 Conclusion

In this paper, we have studied competitive equilibria in generalized matching problems, which include housing markets and roommate and marriage problems as special cases. We have shown that every competitive matching is in the core and that, if there is a competitive matching, then the core is a singleton consisting of the competitive matching. By showing that singleton cores are necessary for the existence of competitive equilibria, we provide another case for the importance of singleton cores in matching problems. We have further shown that a competitive matching exists if and only if the top trading cycles algorithm produces a feasible matching, generalizing the existing result on housing markets. Our results are not only of theoretical interest but also of practical importance. Our results suggest that, in matching problems where a competitive equilibrium exists, decentralized price-taking behavior and cooperative agreement yield the same outcome. A direction for future research is to design a price adjustment process to reach the competitive matching and to compare its informational burden with that of centralized matching algorithms.

Finally, another direction for future research is to study large matching markets. In general, large markets are known to provide a more favorable environment for the existence of competitive equilibria and equivalence between the core and the set of competitive allocations. In our model, existence guarantees equivalence. Hence, it will be interesting to examine the likelihood of the existence of competitive matchings in large matching markets. Our numerical results show that large marriage problems with random preferences tend not to have a competitive matching. An alternative scenario we may analyze to obtain a positive result in the context of bilateral matching problems with random preferences is one where the number of goods is fixed independently of the number of agents while each agent's endowment is randomly drawn from the fixed set of goods.

Acknowledgements

I was introduced to the topic through the research of Joe Ostroy whose continued advice and encouragement are greatly appreciated. I am also grateful to three anonymous referees and participants at Conference on Economic Design 2015 for helpful comments and to the Department of Economics at UCLA for its hospitality during my visit.

References

[1] Abdulkadiroğlu, A. and T. Sönmez (2003), "School Choice: A Mechanism Design Approach," *American Economic Review* 93, 729–747.

[2] Abdulkadiroğlu, A. and T. Sönmez (2013), "Matching Markets: Theory and Practice," *Advances in Economics and Econometrics* 1, Tenth World Congress, 3–47.

[3] Alcalde, J. (1995), "Exchange-Proofness or Divorce-Proofness? Stability in One-Sided Matching Markets," *Economic Design* 1, 275–287.

[4] Ashlagi, I., Y. Kanoria, and J. D. Leshno (2015), "Unbalanced Random Matching Markets: The Stark Effect of Competition," *Working Paper.*

[5] Banerjee, S., H. Konishi, and T. Sönmez (2001), "Core in a Simple Coalition Formation Game," *Social Choice and Welfare* 18, 135–153.

[6] Bikhchandani, S. and J. W. Mamer (1997), "Competitive Equilibrium in an Exchange Economy with Indivisibilities," *Journal of Economic Theory* 74, 385–413.

[7] Chung, K.-S. (2000), "On the Existence of Stable Roommate Matchings," *Games and Economic Behavior* 33, 206–230.

[8] Clark, S. (2006), "The Uniqueness of Stable Matchings," *Contributions in Theoretical Economics* 6, 1–28.

[9] Crawford, V. P. and E. M. Knoer (1981), "Job Matching with Heterogenous Firms and Workers," *Econometrica* 49, 437–450.

[10] Edgeworth, F. Y. (1881), *Mathematical Psychics*, Kegan Paul, London.

[11] Eeckhout, J. (2000), "On the Uniqueness of Stable Marriage Matchings," *Economics Letters* 69, 1–8.

[12] Ehlers, L. and J. Massó (2007), "Incomplete Information and Singleton Cores in Matching Markets," *Journal of Economic Theory* 136, 587–600.

[13] Gale, D. and L. S. Shapley (1962), "College Admissions and the Stability of Marriage," *American Mathematical Monthly* 69, 9–15.

[14] Gudmundsson, J. (2014), "When do Stable Roommate Matchings Exist? A Review," *Review of Economic Design* 18, 151–161.

[15] Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2013), "Stability and Competitive Equilibrium in Trading Networks," *Journal of Political Economy* 121, 966–1005.

[16] Hungerford, T. W. (1974), *Algebra*, Graduate Texts in Mathematics 73, Springer-Verlag, New York.

[17] Kelso, A. S. and V. P. Crawford (1982), "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica* 50, 1483–1504.

[18] Kesten, O. (2006), "On Two Competing Mechanisms for Priority-Based Allocation Problems," *Journal of Economic Theory* 127, 155–171.

[19] Ma, J. (1994), "Strategy-Proofness and the Strict Core in a Market with Indivisibilities," *International Journal of Game Theory* 23, 75–83.

[20] Ma, J. (2002), "Stable Matchings and the Small Core in Nash Equilibrium in the College Admissions Problem," *Review of Economic Design* 7, 117–134.

[21] Moulin, H. (1995), *Cooperative Microeconomics: A Game-Theoretic Introduction*, Princeton Univ. Press, Princeton, NJ.

[22] Niederle, M. and L. Yariv (2009), "Decentralized Matching with Aligned Preferences," *NBER Working Paper* 14840.

[23] Pathak, P. A. and T. Sönmez (2008), "Leveling the Playing Field: Sincere and Sophisticated Players in the Boston Mechanism," *American Economic Review* 98, 1636–1652.

[24] Quinzii, M. (1984), "Core and Competitive Equilibria with Indivisibilities," *International Journal of Game Theory* 13, 41–61.

[25] Rodrigues-Neto, J. A. (2007), "Representing Roommates' Preferences with Symmetric Utilities," *Journal of Economic Theory* 135, 545–550.

[26] Romero-Medina, A. and M. Triossi (2013), "Acyclicity and Singleton Cores in Matching Markets," *Economics Letters* 118, 237–239.

[27] Roth, A. E. (1982a), "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research* 7, 617–628.

[28] Roth, A. E. (1982b), "Incentive Compatibility in a Market with Indivisibilities," *Economics Letters* 9, 127–132.

[29] Roth, A. E. (2007), "Repugnance as a Constraint on Markets," *Journal of Economic Perspectives* 21, 37–58.

[30] Roth, A. E. and E. Peranson (1999), "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review* 89, 748–780.

[31] Roth, A. E. and A. Postlewaite (1977), "Weak Versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics* 4, 131–137.

[32] Shapley, L. S. and H. E. Scarf (1974), "On Cores and Indivisibility," *Journal of Mathematical Economics* 1, 23–37.

[33] Shapley, L. S. and M. Shubik (1971), "The Assignment Game I: The Core," *International Journal of Game Theory* 1, 111–130.

[34] Sönmez, T. (1996), "Implementation in Generalized Matching Problems," *Journal of Mathematical Economics* 26, 429–439.

[35] Sönmez, T. (1999), "Strategy-proofness and Essentially Single-Valued Cores," *Econometrica* 67, 677–689.

[36] Sotomayor, M. (2007), "Connecting the Cooperative and Competitive Structures of the Multiple-Partners Assignment Game," *Journal of Economic Theory* 134, 155–174.

[37] Tan, J. J. M. (1991), "A Necessary and Sufficient Condition for the Existence of a Complete Stable Matching," *Journal of Algorithms* 12, 154–178.

[38] Voorneveld, M. and H. Norde (1997), "A Characterization of Ordinal Potential Games," *Games and Economic Behavior* 19, 235–242.