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INCENTIVE COMPATIBILITY ON THE DOMAIN OF SINGLE-PEAKED PREFERENCES

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ABSTRACT. We consider incentive compatible voting rules on the domain of single-peaked preferences. In the environment where the value distributions are generic in the set of independent beliefs, we show that every incentive compatible rule satisfies the tops-only property.

Keywords: Incentive compatibility, Single-peaked preferences, Tops-only property

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1. Introduction

We study incentive compatibility of voting rules on the domain of single-peaked preferences. Incentive compatible voting rules require agents to be truth-telling if other agents are also truth-telling when there is incomplete information about other agent's preferences. The domain of single-peaked preferences is a popular restricted domain of preferences. Roughly speaking, suppose that the alternatives can be ordered in some way (left to right, smallest to biggest, etc). We say that agents have single-peaked preferences over the alternatives if each agent has the most preferred alternative (the top) and the alternative closer to the top is preferred more.

We consider the generic environment introduced in Majumdar and Sen (2004). On the domain of unrestricted preferences, they show that any incentive compatible rule is dictatorial.[1] But, on a domain of restricted preferences, particularly single-peaked preferences, this result does not hold. Moulin (1980) shows that even a rule that satisfies a stronger condition than incentive compatibility is not necessarily dictatorial. We build on these result by deriving a new implication for incentive compatible rules on the domain of single-peaked preferences. We show that any incentive compatible rule satisfies the tops-only property, i.e., the rule depends only on the tops in agents' preferences.

2. The Model

Consider a standard Bayesian environment with private values. The set of agents is $N = \{1,2,...,n\}$ and the set of alternatives is $L = \{1,2,...,m\}$ with $m \geq 3$. Each agent $i \in N$ has a value function for each alternative $v_i : L \to \mathbb{R}$. We denote by $v_i = (v_i(l))_{l\in L} \in \mathbb{R}^m$ the vector of values that agent i assigns to the alternatives. We consider single-peaked preferences where the value set, V_i over alternatives has the following property.

There exists an alternative $l_i^* \in L$, the top of v_i, such that for every $v_i \in V_i$ and $l, l' \in L$,

$$\begin{cases} l' < l \leq l_i^* & \Rightarrow v_i(l') < v_i(l) \\ l_i^* \leq l' < l & \Rightarrow v_i(l') > v_i(l) \end{cases}$$

The set of value profiles is denoted by $V = V_1 \times \cdots \times V_n$. We assume that the distribution over V has full support.

The set of orderings of alternatives is T^{ORD}, and let t denote an element of T^{ORD}. We define an ordinal type function $t_i : V_i \to T^{ORD}$, which shows the ordinal preferences of agent i. On the domain of single-peaked preferences, the associated type set is T, a subset of the

[1]In Section 2, we see the equivalence between the incentive compatibility in our set-up and their concept of incentive compatibility. Also the generic environment is formally defined.

ordinal type set, i.e., $T \subset T^{ORD}$. The full support assumption implies the *full* single-peaked preferences domain. In other words, it allows all admissible ordinal preferences satisfying the above property. For example, with 3 alternatives, $T = \{123, 213, 231, 321\} \subset T^{ORD} = \{123, 132, 213, 231, 312, 321\}$.

A voting rule is a mapping , $g : T^n \to L$, which means that we focus on deterministic, and ordinal rules that depend only on ranking information of preferences. We mainly consider unanimous rules that in any case where all agents agree on a particular alternative as the best, then the rules must choose that alternative. This is a commonly-used axiom in the social choice literature. It is useful to define another function related to ranking of alternatives. For $i \in N$, let $r_k : T \to L$ denote the kth-ranked alternative in t_i.

Definition 1. A rule g is unanimous if $g(t_i, t_{-i}) = l$ whenever $l = r_1(t_i)$ for every agent $i \in N$

The information structure is the following. Since we deal with ordinal rules, we use the induced distribution over type profiles, $\mu(t_1, ..., t_n) = pr(\{v \in V : t_i(v_i) = t_i \; for \; every \; i \in N\})$. For any $Q \subseteq T^n$, let $\mu(Q) = \sum_{(t_1,...,t_n) \in Q} \nu(t_1, ..., t_n)$. Majumdar and Sen (2004) introduce the set of distributions which satisfy the following two conditions. First, types are independent across agents. Second, μ satisfies the following property: for all $Q, R \subset T^n$,

$$[\mu(Q) = \mu(R)] \Rightarrow [Q = R].$$

We denote this set of distributions by C and the set of all independent distributions by Δ^I. They say that C is large and generic in Δ^I with the two properties: (1) C is open and dense in Δ^I and (2) $\Delta^I - C$ has Lebesgue measure zero. Similarly, we also consider the generic environment, $\mu \in C$.

We define incentive compatibility in our environment.

Definition 2. A rule g is Incentive Compatible (IC) if for all $i \in N$, for all $t_i, t_i' \in T$, for all v_i such that $t_i(v_i) = t_i$, we have

$$\sum_{t_{-i} \in T^{n-1}} v_i(g(t_i, t_{-i})) \, \mu(t_i, t_{-i} \mid t_i) \geq \sum_{t_{-i} \in T^{n-1}} v_i(g(t_i', t_{-i})) \, \mu(t_i, t_{-i} \mid t_i)$$

Since we assume that V has a full support and rules are ordinal, our incentive compatibility is equivalent to ordinally Bayesian incentive compatibility in Majumdar and Sen (2004).[2] They prove that given $\mu \in C$, if a rule g is unanimous and IC then g is dictatorial on the unrestricted domain. But, the following example shows that there exists a unanimous, IC but a non-dictatorial rule even with $\mu \in C$ if we consider the domain of *single-peaked preferences*.

[2]We discuss this equivalence in Section 4.

Example 1. Consider the rule g defined below with three alternatives and two agents.

$t_2 \diagdown t_1$	123	213	231	321
123	1	1	1	1
213	1	2	2	2
231	1	2	2	2
321	1	2	2	3

In the array, agent 1's types appear along the rows and agent 2's along the columns. The inside entries indicate $g(t_1, t_2)$. Observe that the rule is unanimous and non-dictatorial. In fact, the rule is IC with respect to any distribution, because the probabilities that each alternative is chosen under the rule conditional on a truthful type announcement first-order stochastically dominate the probabilities conditional on other type announcements regardless of any distribution.

3. Result

We introduce the tops-only property of a rule and and show the main result.

Definition 3. A rule g satisfies the tops-only property if $g(t_1, ..., t_n) = g(t'_1, ..., t'_n)$ whenever $(t_1, ..., t_n)$ and $(t'_1, ..., t'_n)$ are such that $r_1(t_i) = r_1(t'_i)$ for all $i \in N$.

Theorem. *Given $\mu \in C$, every unanimous and IC rule g satisfies the tops-only property.*

Proof of Theorem:

The first step is to show a useful lemma that given $\mu \in C$, an IC rule g should satisfy a certain property called Property M.[3] The second step is to prove that if g satisfies the unanimity and Property M, then g has the tops-only property.

Step 1) For the lemma, we define the set of alternatives that are weakly preferred to alternative l of type t_i (the upper contour set) such that $B(l, t_i) = \left\{ l' \in L : v_i^{l'} \geq v_i^l \; for \, t(v_i) = t_i \right\}$. Also, we define Property M below.

Definition 4. A rule g satisfies Property M, if for every agent $i \in N$, for all integers $k = 1, ..., m$, and for every t_i, t'_i such that $B(r_k(t_i), t_i) = B(r_k(t'_i), t'_i)$, we have

$[g(t_i, t_{-i}) \in B(r_k(t_i), t_i)] \Rightarrow [g(t'_i, t_{-i}) \in B(r_k(t'_i), t'_i)]$

[3]We borrow the name and definition of this property from Majumdar and Sen (2004).

It implies that when two ordinal types of agent i, t_i and t_i' share the best k alternatives and $g(t_i,t_{-i})$ is one of the best k alternatives, Property M demands that $g(t_i',t_{-i})$ must be one of the best k alternatives.

Lemma. *Given $\mu \in C$, an IC rule g satisfies Property M.*

The proof of the lemma is omitted because it is shown in the proof of Theorem 4.1 in Majumdar and Sen (2004)

Step 2) Since the formal proof has some technical parts with arbitrary number of agents and alternatives, we place it in the Appendix. Instead, we provide a simple example that shows the crucial idea of the second step.

Example 2. Let $L = \{1,2,3\}$, $N = \{1,2\}$. Given $\mu \in C$, consider an incomplete rule g below.

$t_2 \diagdown t_1$	123	213	231	321
123	1	l	l'	1
213		2	2	
231		2	2	
321	1			3

First, g is unanimous. Assume that g is IC, but does not satisfy the tops-only property. Then, there exists (t_1,t_2) and (t_1',t_2') with $r_1(t_i) = r_1(t_i')$ for $i = 1$ or 2 such that $g(t_1,t_2) \neq g(t_1',t_2')$. Consider $(t_1,t_2) = (213,123)$ and $(t_1',t_2') = (231,123)$ and let $g(213,123) = l$ and $g(231,123) = l'$ (For other empty entries, the argument is similar.) If l or l' is equal to 2 $(= r_1(t_1) = r_1(t_1'))$, the lemma requires that $l = l' = 2$, which contradicts that $l \neq l'$. Hence, l or l' should be 3 because $l, l' \neq 2$ and $l \neq l'$. But, it also leads a contradiction that the lemma requires that l or l' should be 1 or 2 because $B(r_2(123), 123) = B(r_2(213), 213)$ and $g(213,213) = g(231,213) = 2$ from the unanimity of the rule. The formal proof extends this idea with the arbitrary number of agents and alternatives.□

Theorem implies that when we design an IC rule we can reduce our attention to the set of rules that satisfy the tops-only property. In terms of information to implement IC rules, the information about the most preferred alternatives of all agents, not about full ranking over alternatives is sufficient.

4. DISCUSSION

The seminal Gibbard-Satterthwaite theorem (Gibbard (1973) and Satterthwaite(1975)) studies strategy-proof voting rules in which truth-telling is best for each agent regardless of other agents' reports. It states that under mild assumptions, a deterministic rule that is strategy-proof is dictatorial. Majumdar and Sen (2004) analyze the implications of weakening incentive constraints from strategy proofness to Ordinal Bayesian Incentive Compatibility, which is similar to our approach. While they focus on the unrestricted domain, we divert our attention to the domain of single-peaked preferences and show the relationship of incentive compatibility and the tops-only property.

Regarding the tops-only property, Weymark (2008) shows on the domain of single-peaked preferences that every unanimous and strategy-proof rule satisfies the tops-only property. Our result confirms that weakening the strategy-proofness to incentive compatibility with respect to a generic independent prior does not affect the result in Weymark (2008). Furthermore, Chatterji and Sen (2011) investigate the characteristics of domains that induce the property that every unanimous and strategy-proof rule satisfies the tops-only property. The similar investigation with weakening strategy-proofness could be interesting and our result would be a particular case.

Our result implies that a unanimous and ordinally Bayesian IC rule satisfies the tops-only property because the equivalence relationship in our environment. But, the relationship may not hold if we do not assume the full support of value distributions or ordinal rules, allowing cardinal rules. Thus, the study of IC rules without our assumptions could be an interesting direction in future.

With two alternatives, Azrieli and Kim (2014) investigate Pareto efficiency and incentive compatibility of rules. In this paper, we regard unanimity as a concept of efficiency. Further study regarding other concepts of efficiency and incentive compatibility of rules may be interesting.

APPENDIX

Step 2 in the proof of Theorem. Assume that a rule g is unanimous and IC, but does not satisfy the tops-only property. It implies that there exists two type profiles (t_i, t_{-i}) and (t_i', t_{-i}) with $r_1(t_i) = r_1(t_i')$ such that $g(t_i, t_{-i}) \neq g(t_i', t_{-i})$. Without loss of generality, fix the agent i as agent 1 and consider t_1, t_1' such that $t_1 \neq t_1'$, $r_1(t_1) = r_1(t_1') = l^*$, and $r_k(t_1) = r_k(t_1')$ for $k = \{1, ..., l-1, l+2, ..., m\}$. In other words, all the ranked alternatives of two types t_1 and t_1'

are the same except the l th and $l+1$ th-ranked alternatives. Note that $r_l(t_1) > l^* > r_{l+1}(t_1)$ or $r_l(t_1) < l^* < r_{l+1}(t_1)$ because of the domain of single-peaked preferences. We can think of three cases according to $g(t_1, t_{-1})$.

Case I: $g(t_1, t_{-1}) = l^*$. By the lemma, $g(t_1', t_{-1}) = l^*$, which contradicts that $g(t_1, t_{-1}) \neq g(t_1', t_{-1})$. Thus, $g(t_1, t_{-1}) \neq l^*$.

Case II: $g(t_1, t_{-1}) > l^*$. Then, $g(t_1', t_{-1}) < l^*$. Suppose not, $g(t_1', t_{-1}) \geq l^*$. If $g(t_1', t_{-1}) = l^*$, similarly to Case I, $g(t_1, t_{-1}) = l^*$, contradiction. If $g(t_1', t_{-1}) > l^*$ and without loss of generality $g(t_1, t_{-1}) < g(t_1', t_{-1})$, then the single-peaked preferences domain guarantees that there exists k such that $B(r_k(t_1), t_1) = B(r_k(t_1'), t_1')$ and $g(t_1, t_{-1}) \in B(r_k(t_1), t_1)$, but $g(t_1', t_{-1}) \notin B(r_k(t_1'), t_1')$. It contradicts that g satisfies Property M.

Case III: $g(t_1, t_{-1}) < l^*$. Then $g(t_1', t_{-1}) > l^*$ because of the analogous argument in Case II.

Thus, we can start with that $g(t_1, t_{-1}) > l^* > g(t_1', t_{-1})$ or $g(t_1, t_{-1}) < l^* < g(t_1', t_{-1})$. After the following argument with agent 2, we will apply the same argument to agent 3,...n and show the contradiction to the unanimity of the rule.

Consider the case that $g(t_1, t_{-1}) > l^* > g(t_1', t_{-1})$ (In the other case, the contradiction can be similarly shown). Take another agent, say agent 2. We have three cases according to $r_1(t_2)$

Case A: $r_1(t_2) < l^*$, then the full single-peaked preferences domain guarantees that there exists t_2' such that $r_1(t_2') = l^*$ and, for some k, $B(r_k(t_2), t_2) = B(r_k(t_2'), t_2')$, $g(t_1, t_2, ..., t_n) \notin B(r_k(t_2), t_2)$ but $l^* \in B(r_k(t_2), t_2)$. By the lemma, $g(t_1, t_2', ..., t_n) \neq l^*$(Suppose not,$g(t_1, t_2, ..., t_n) \notin B(r_k(t_2), t_2)$.)

Case B: $r_1(t_2) = l^*$, set $t_2' = t_2$.

Case C: $r_1(t_2) > l^*$, then the full single-peaked preferences domain guarantees that there exists t_2' such that $r_1(t_2') = l^*$ and, for some k, $B(r_k(t_2), t_2) = B(r_k(t_2'), t_2')$, $g(t_1', t_2, ..., t_n) \notin B(r_k(t_2), t_2)$ but $l^* \in B(r_k(t_2), t_2)$. By the lemma, $g(t_1', t_2', ..., t_m) \neq l^*$.

Therefore, at least one of $g(t_1, t_2', ..., t_n)$ or $g(t_1', t_2', ..., t_n)$ cannot be l^*. From the arguments of Case I,II, and III we have that $g(t_1, t_2', ..., t_m) > l^* > g(t_1', t_2', ..., t_m)$ or $g(t_1, t_2', ..., t_n) < l^* < g(t_1', t_2', ..., t_n)$.

Again, we can fix t_2' and apply the same argument to agent 3 and repeat the same procedure to agent 4,..., agent$n$. Then we reach that $g(t_1, t_2', t_3', ..., t_n') > l^* > g(t_1', t_2', ..., t_n')$ or $g(t_1, t_2', t_3', ..., t_n') < l^* < g(t_1', t_2', t_3', ..., t_n')$ with $r_1(t_i') = l^*$ for every $i \in N$, which contradicts the unanimity of the rule.□

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