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Abstract: Given the importance of uncertainty shocks to economic agents such as consumers, producers, investors, or policymakers, it seems surprising that little attention has been paid to developing models analyzing the effect of uncertainty (second-moment) shocks. In contrast, there have been a vast amount of literature dealing with the impact of level (first-moment) shocks. In this paper, we attempt to fill this gap by proposing a new concept: 'generalized impulse.' This concept is defined as a one-off external intervention given to a system which results in a change in the distribution of the internal structural errors of the system. Such an intervention can be given to the system in order to achieve some policy objectives, or it can be given exogenously by an external force outside the system. Uncertainty shocks are generated as a special case of such a generalized impulse. We also propose new impulse response functions called 'variance impulse response function' and 'covariance impulse response function,' which can enable researchers to measure the impact of uncertainty shocks. We then apply our new methods to analyze the impact of uncertainty shocks in oil prices on the GDP growth rate, using data from the United States. When the level of uncertainty in oil prices unexpectedly increases, the price of oil tends to increase significantly and persistently, whereas the growth rate of GDP is adversely affected. Such a negative impact on GDP is present even after five years. Hence, our results indicate that an unexpected increase in uncertainty in oil prices can have an effect similar to an unexpected increase in oil prices themselves, but in a much worse manner. This is because the negative impact on output induced by uncertainty shocks can be much more persistent than the impact from level shocks.

Keywords: Generalized Impulse, Uncertainty Shocks, Level Shocks, Intervention, Variance Impulse Response Function

JEL classifications: C32, C54, E52.

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1 Introduction

Since the seminal paper by Sims (1980), impulse response analysis has become one of the most widely used statistical methods in many areas of economics, business, and finance. In particular, it is an indispensable toolkit in macroeconomics for analyzing policy effects. As an alternative to simultaneous equation models of structural form, which can require problematic identification conditions, vector autoregressive (VAR) models have been put forward for use in analyzing macroeconomic variables. Due to their effectiveness, VAR models have been widely adopted in areas such as business and finance. VAR models are powerful in terms of forecasting variables of interest. Once an appropriate number of lagged variables is determined, predictions for any desired forecast horizon can be made effortlessly.

However, as the number of variables and the number of lags increase, it becomes increasingly difficult to extract or interpret meaningful relationships between the variables used in VAR models. Impulse response functions (IRF) are an efficient way of summarizing the potentially complicated relationships among the variables used in VAR models. If we have a VAR model that mimics the real economy and a suitable identification scheme is in place, then IRFs can describe how the effect of an external intervention on some variables in the system evolves throughout some period of future time. In other words, IRFs can tell us how the impact of a shock propagates throughout the whole system through time. This information is precisely what any policymaker wishes to know. There have been many successful applications of the IRF methodology in the literature.[1]

In this paper, we discuss two issues. The first is what we mean by 'impulse.' The second

[1] A comprehensive and insightful discussion on all relevant literature regarding the SVAR modeling methodology is presented in Kilian and Lutkepohl (2017).

is how to measure its impact. To start, we look at the definition of 'impulse' in The New Palgrave Dictionary of Economics by Lutkepohl (2008): "*In this framework impulse response analysis may be based on the counterfactual experiment of tracing the marginal effect of a shock to one variable through the system by setting one component of* ε_t *to one and all other components to zero and evaluating the responses of the* y_t*'s to such an impulse as time goes by.*" Here, ε_t can be viewed as a vector of structural errors vibrating the system randomly at each time t. Hence, setting one component of ε_t (i.e., the i^{th} component of ε_t denoted ε_{it}) to 1 and all other components to zero means that there is a sudden mean shift only in one structural error. To generalize the method, we can set ε_{it} to some non-zero scalar number δ and all other components to zero. In that case, the shifted mean of structural error ε_{it} is simply δ instead of 1. This is the typical and conventional way to define the term 'impulse' in the literature on impulse response analysis.

In our framework, an impulse is a one-time only external intervention given by system managers or policymakers to the system, which results in a change in the distribution of structural errors. To clarify the idea, consider a situation where an impulse is applied to ε_{it} and as a result, structural error ε_{it} changes to a different structural error $\tilde{\varepsilon}_{it}$. For such an intervention to be effective, the distribution of $\tilde{\varepsilon}_{it}$ must be different from the distribution of ε_{it}. The conventional impulse can be expressed as a special case; that is, $\tilde{\varepsilon}_{it} = \varepsilon_{it} + \delta$. Hence, the distribution of $\tilde{\varepsilon}_{it}$ is simply obtained by shifting the distribution of ε_{it} by δ. This kind of impulse is called a 'mean shift impulse,' a 'location shock,' or a 'level shock.'

It is hard to accept that system managers or policymakers are interested only in manipulating the location of the distribution of structural errors of interest. System managers or policymakers might also be interested in manipulating other characteristics of the distribution, such as the level of uncertainty (variance or second-moment), as well as higher moments like skewness or kurtosis.

Suppose that a one-off intervention is imposed on the system to reduce the level of uncertainty. Then, the intervention results in a new distribution of $\tilde{\varepsilon}_{it}$ with a smaller variance than the original distribution of ε_{it}. Specifically, it means that $\tilde{\varepsilon}_{it} = \lambda\varepsilon_{it}$ where $\lambda \in (0, 1)$; that is, a reduced level of uncertainty without any mean shift. Such an intervention is called an 'uncertainty shock' or a 'second-moment shock.' In our framework, an impulse is defined as an external intervention given to a system to manipulate the distribution of internal structural errors, which we refer to as a 'generalized impulse.'

We wish to emphasize that our intention is not to generalize impulse response functions, but rather to expand the concept of impulse itself. The notion of generalized impulse response functions was initially introduced by Koop et al. (1996) and subsequently developed in a series of papers, including Potter (2000). This concept extends the traditional mean impulse response functions in order to make them applicable in nonlinear models. However, it is imperative to clarify that these advancements do not modify the core concept of impulse itself. The conventional mean shift impulse framework remains a ubiquitous fixture across the spectrum of all relevant research papers to date.

Bloom (2009) correctly points out: "*Despite the size and regularity of these second-moment (uncertainty) shocks, there is no model that analyzes their effects. This is surprising given the extensive literature on the impact of first-moment (levels) shocks. This leaves open a wide variety of questions on the impact of major macroeconomic shocks, since these typically have both a first- and a second-moment component.*" Our paper can be seen as an extension of this idea to impulse response analysis.

The second issue we discuss in this paper is how to measure the propagation mechanism of such a generalized impulse within the structural vector-autoregressive (SVAR) framework. Considering the conventional impulse is commonly represented as a mean shift in the literature, it is natural to measure its impact using the conditional mean function, which has been predominantly employed in most papers. Although the concept of 'generalized impulse' is presented in this paper for the first time, there are a few studies in the literature that attempt to measure the response of variables to a conventional mean shift impulse in a non-standard way, i.e., not relying on the conditional mean function. White et al. (2015), Chavleishvili and Manganelli (2016), Han et al. (2019), Montes-Rojas (2019), and Lee et al. (2021) propose a novel measure called the 'quantile impulse response function,' which tracks how variables respond to a mean shift impulse by evaluating the conditional quantile function. They empirically discover there are significant fluctuations in the quantile impulse response function even in response to a mean shift impulse alone. However, those papers do not answer a potentially more interesting question: Why does a mean shift impulse generate ripples not only in the conditional mean function, but also in the conditional quantile function? In this paper, we not only provide an answer to that question but we also discuss how to evaluate the impact of generalized impulses, such as uncertainty shocks, using a new measure called the 'variance impulse response function.'

The rest of this paper is organized as follows. In Section 2, we review the conventional impulse response functions. We formally introduce the concept of 'generalized impulse' and discuss how to measure its impact in Section 3. In Section 4, we consider a special case of the generalized impulse in which an external intervention is given to a system to manipulate the level of uncertainty generating uncertainty shocks within that system. Section 5 derives response functions based on our generalized impulse framework, and discusses the so-called cross-over effect in general models, and some numerical analysis is performed in Section 6. An empirical example is given in Section 7 and section 8 provides concluding remarks.

2 Review on Conventional Impulse Response Functions

Although impulse response functions can easily be derived from fairly complex data generating processes, we will consider the simplest bivariate data generating process. This section does not contain any new information. However, we provide a detailed presentation to establish our core concepts and introduce the notations that will be utilized in the following sections. Let us consider a sequence of endogenous variables denoted $\{..., y_{t-2}, y_{t-1}, y_t, y_{t+1}, y_{t+2}, ...\}$ where y_t is a 2×1 vector given by $y_t = (y_{1t}, y_{2t})'$. We consider the SVAR model of order 1 for y_t as follows:

$$A_0 y_t = c + A_1 y_{t-1} + \varepsilon_t \tag{1}$$

where A_0 and A_1 are 2×2 coefficient matrices, c is a 2×1 intercept vector, and $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ is the 2×1 vector of mean-zero structural error terms with a diagonal variance-covariance matrix denoted by $\Sigma = diag(\sigma_1^2, \sigma_2^2)$.

We will assume that (i) the diagonal elements of contemporaneous coefficient matrix A_0 are normalized to 1; (ii) the stationarity condition on (1) is satisfied, i.e., all roots of w satisfying $|A_0 - A_1 w| = 0$ lie outside the unit circle; and (iii) a sufficient set of identification conditions is imposed on population parameter matrices; A_0, A_1, Σ. We note that the usual VAR forecasting error (denoted by u_t) is given by $u_t = A_0^{-1}\varepsilon_t$. Since the main point of this paper is not to propose a new identification scheme but to propose a new concept of impulse, we will simply employ the usual Cholesky identification scheme; i.e., A_0 is a lower triangular matrix. Under such a particular identification restriction, endogenous variable y_{2t} does not have a contemporaneous impact on y_{1t}. For example, y_{1t} and y_{2t} can be the GDP growth rate and the policy interest rate of an economy, respectively. Therefore, error term ε_{2t} in such an example can be interpreted as a monetary policy shock. One might be interested in understanding how the effect of an external intervention to the monetary policy shock (ε_{2t}) on the GDP growth rate (y_{1t}) evolves through time. It might be helpful to have this example in mind to follow our subsequent discussion.

As time flows from the infinite past to the infinite future, we will assume there is one-off unexpected intervention to the structural error term ε_{2t} occurring only at the present time (denoted by t). Hence, in our notation, $t - i$ represents an earlier time while $t + j$ represents a future time for any positive integers i and j. The term 'unexpected intervention' is vague at the moment, and needs to be precisely stated and modeled. In the literature on impulse response functions, the term is exclusively defined as unexpectedly changing error term ε_{2t} to another error $\tilde{\varepsilon}_{2t}$ at time t, which is defined as $\tilde{\varepsilon}_{2t} = \varepsilon_{2t} + \delta$; i.e., a mean shift intervention. Here, the size of δ is viewed as the magnitude of the one-off intervention.

The time path of y_{1t} without the intervention would obviously be

$$\{..., y_{1t-2}, y_{1t-1}, y_{1t}, y_{1t+1}, y_{1t+2}, ...\},$$

whereas the time path of the observable with the intervention is

$$\{..., y_{1t-2}, y_{1t-1}, \tilde{y}_{1t}, \tilde{y}_{1t+1}, \tilde{y}_{1t+2}, ...\},$$

where the observables affected by the intervention is denoted $\tilde{y}$. Hence, the one-off intervention at time t will have a lasting effect on the observable. The two time paths start to diverge at time t, and evolve under different dynamics for a while although such a difference will eventually die out under certain stationary conditions.

The conventional impulse response function is often denoted by

$$\frac{\partial y_{t+s}}{\partial \varepsilon_t'} = \begin{bmatrix} \frac{\partial y_{1t+s}}{\partial \varepsilon_{1t}} & \frac{\partial y_{1t+s}}{\partial \varepsilon_{2t}} \\ \frac{\partial y_{2t+s}}{\partial \varepsilon_{1t}} & \frac{\partial y_{2t+s}}{\partial \varepsilon_{2t}} \end{bmatrix}, \tag{2}$$

where $s = 0, 1, ..., h$ for forecast horizon h. The SVAR(1) model in (1) can be converted into an MA(∞) process given by

$$y_{t+s} = \mu + \Phi_0 \varepsilon_{t+s} + \Phi_1 \varepsilon_{t+s-1} + ... + \Phi_{s-1}\varepsilon_{t+1} + \Phi_s \varepsilon_t + \Phi_{s+1}\varepsilon_{t-1} + \Phi_{s+2}\varepsilon_{t-2}..., \tag{3}$$

where $\mu = (I - A_0^{-1}A_1)^{-1}A_0^{-1}c$ and $\Phi_i = (A_0^{-1}A_1)^i A_0^{-1}$. Therefore, it is easily obtained that

$$\frac{\partial y_{t+s}}{\partial \varepsilon_t'} = \Phi_s.$$

Note that the effect of the intervention in ε_{2t} on y_{1t+s} (denoted by $\frac{\partial y_{1t+s}}{\partial \varepsilon_{2t}}$) is captured by Φ_{s12}; the (1, 2)th element of $\Phi_s = \begin{bmatrix} \Phi_{s11} & \Phi_{s12} \\ \Phi_{s21} & \Phi_{s22} \end{bmatrix}$.

Therefore, such a derivative measures the effect of the unexpected change in ε_t at time t on the future values of observables y_{t+s} up to the pre-specified forecast horizon h. Alternatively, such an effect can be measured as the difference between two conditional expectations. Let us define $\mathcal{F}_{t-1}$ as the set of all information available at time $t-1$ including $\{\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...\}$. In line with White et al. (2015) and Lee et al. (2021), we use an equivalent, but more rigorous, definition of the conventional impulse response function, called the 'mean impulse response function (MIRF),' as follows:

$$\Delta_E(s) = E(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) - E(y_{t+s}|\mathcal{F}_{t-1}), \tag{4}$$

where $s = 0, 1, ..., h$. To obtain the first term in (4), we note that the affected y_{t+s} (i.e., $\tilde{y}_{t+s}$) is derived from (3) by simply replacing ε_t with $\tilde{\varepsilon}_t = (\varepsilon_{1t}, \tilde{\varepsilon}_{2t})'$ as follows:

$$\tilde{y}_{t+s} = \mu + \Phi_0 \varepsilon_{t+s} + \Phi_1 \varepsilon_{t+s-1} + ... + \Phi_{s-1} \varepsilon_{t+1} + \Phi_s \tilde{\varepsilon}_t + \Phi_{s+1} \varepsilon_{t-1} + ..., \tag{5}$$

from which we obtain

$$\begin{aligned} E(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) &= \mu + \Phi_s \begin{bmatrix} 0 \\ \delta \end{bmatrix} + \Phi_{s+1}\varepsilon_{t-1} + \Phi_{s+2}\varepsilon_{t-2} + ... \\ &= \mu + \begin{bmatrix} \delta\Phi_{s12} \\ \delta\Phi_{s22} \end{bmatrix} + \Phi_{s+1}\varepsilon_{t-1} + \Phi_{s+2}\varepsilon_{t-2} + ... \end{aligned}$$

The second term in (4) is directly obtained from (3) as follows:

$$E(y_{t+s}|\mathcal{F}_{t-1}) = \mu + \Phi_{s+1}\varepsilon_{t-1} + \Phi_{s+2}\varepsilon_{t-2} + ...$$

As a result, the MIRF is

$$\Delta_E(s) = \begin{bmatrix} \delta\Phi_{s12} \\ \delta\Phi_{s22} \end{bmatrix}, \tag{6}$$

and the effect of the intervention in ε_{2t} on y_{1t} is captured by $\delta\Phi_{s12}$, which is identical to $\frac{\partial y_{1t+s}}{\partial \varepsilon_{2t}}$ if the magnitude of the intervention (δ) is set to 1. Hence, we can easily see that the definitions in (2) and (4) are equivalent. Since it is possible to have an intervention on ε_{1t} instead of ε_{2t}, we will employ a more general notation, $\delta = (\delta_1, \delta_2)'$, where each element δ_i is an intervention on ε_{it}. With this notation, the MIRF in (6) is written as

$$\Delta_E(s) = (A_0^{-1}A_1)^s A_0^{-1}\delta, \tag{7}$$

where $\delta = (0, \delta_2)'$. We find that the second method in (4) to define response functions is more suitable for generalization. Therefore, we will employ this method to measure the effects of our generalized impulse in the following sections.

3 Generalized Impulse

We observe in the previous section that when a one-off intervention from outside the system is applied to one of the structural errors within the system, the distribution of the error will change; that is,

$$\varepsilon_t \rightarrow \tilde{\varepsilon}_t,$$

where the resulting new structural error, $\tilde{\varepsilon}_t$, follows a different distribution than ε_t. Let $f_\varepsilon(z)$ and $f_{\tilde{\varepsilon}}(z)$ be the probability density functions of ε_t and $\tilde{\varepsilon}_t$, respectively. If we define $\tilde{\varepsilon}_t$ as a mean shifted version of ε_t (i.e., $\tilde{\varepsilon}_t = \varepsilon_t + \delta$) as done exclusively in the literature, then $f_{\tilde{\varepsilon}}(z) = f_\varepsilon(z - \delta)$. However, it is reasonable to argue that an intervention from outside the system can modify not only the mean of the distribution, but can also alter any other characteristics of the underlying distribution. In fact, the entire distribution can be changed.

Policymakers can decide which characteristics of the underlying distribution of a key variable they would like to manipulate. We illustrate this point with an example of monetary policy. If the central bank unexpectedly announces it will increase its policy interest rate, then such an intervention exactly fits the conventional definition of intervention; that is, a mean shift intervention of the form $\tilde{\varepsilon}_t = \varepsilon_t + \delta$. However, raising or lowering the policy rate is not the only tool the central bank can use to combat inflation or stimulate economic activity. The central bank can alter the level of uncertainty that market participants have about future interest rates, even without changing the level of their policy interest rate at all. For example, the central bank can unexpectedly announce it will maintain the current level of its policy rate for some period of time. Intervention is not achieved only through action; sometimes inaction can be more effective. Although such an announcement does not change the benchmark interest rate, it can greatly reduce uncertainty over the course of market interest rates, which can result in boosting economic activity. Such an intervention results in a new distribution, $f_{\tilde{\varepsilon}}(z)$, which has a smaller variance than $f_\varepsilon(z)$ but with the same mean level as $f_\varepsilon(z)$. Hence, in this case, we have $\tilde{\varepsilon}_t = \lambda\varepsilon_t$ where $\lambda \in (0, 1)$. It is also possible to design an intervention that can alter other characteristics of the underlying distribution, such as skewness or kurtosis.

Suppose an external intervention is applied to a system to manipulate the distribution of internal structural error term ε_t within the system. As a result, the original distribution of error term ε_t is changed from $f_\varepsilon(z)$ to $f_{\tilde{\varepsilon}}(z)$ which corresponds to new structural error term $\tilde{\varepsilon}_t$. Such an external intervention is referred to as a 'generalized intervention' or a 'generalized impulse.'

Let y_{t+s} and $\tilde{y}_{t+s}$ be the time paths of observables in the system without and with such a generalized impulse, respectively. The impact of the generalized impulse can be measured by the following distributional impulse response function (DIRF)

$$\Delta_\Upsilon(s) = \Upsilon(f_{\tilde{y}_{t+s}}(\cdot)|\mathcal{F}_{t-1}) - \Upsilon(f_{y_{t+s}}(\cdot)|\mathcal{F}_{t-1})$$

where $s = 0, 1, ..., h$. Here, $\Upsilon(f_X(\cdot)|\mathcal{F}_{t-1})$ is a mapping from a probability density function to some characteristic of the density function. For example, for a mean shift intervention, the obvious choice for the mapping is $\Upsilon(f_X(\cdot)|\mathcal{F}_{t-1}) = E(X|\mathcal{F}_{t-1})$. When the intervention comes in the form of changing uncertainty, then the choice for the mapping can be $\Upsilon(f_X(\cdot)|\mathcal{F}_{t-1}) = Var(X|\mathcal{F}_{t-1})$. In alternative scenarios where an external intervention is designed to manipulate either the skewness or the kurtosis of the underlying distribution, the natural measure should be either the conditional skewness function or conditional kurtosis function, respectively. However, in the following sections, we will focus only on impulse response functions based on the first and second moments.

4 A Special Case: An Impulse That Changes Uncertainty

As mentioned in the previous section, we will consider a situation where an external intervention takes the form of a mean shift or a scale shift. We propose the following definition of generalized intervention

$$\tilde{\varepsilon}_t = \delta + \lambda \varepsilon_t, \tag{8}$$

where $\delta = (\delta_1, \delta_2)$ and $\lambda = diag\{\lambda_1, \lambda_2\}$ in which $\lambda_1 > 0$ and $\lambda_2 > 0$. From now on, we use the *diag* operator in two ways: (i) if its argument is a vector, the outcome is a diagonal matrix, and (ii) if its argument is a square matrix, the outcome will be a vector consisting of all diagonal elements. While δ represents the location shift factor, as in the conventional definition of intervention, the new feature, λ, represents a scaling factor that controls the degree of uncertainty from a policy intervention. The example mentioned in the previous section, announcing no change in the interest rate, can be specified as $\delta = 0$ and $\lambda = diag\{1, \lambda_2\}$ with $\lambda_2 \in (0, 1)$. We note that $E(\tilde{\varepsilon}_t|\mathcal{F}_{t-1}) = \delta$ and $Var(\tilde{\varepsilon}_t|\mathcal{F}_{t-1}) = \lambda\Sigma\lambda'$.

Policymakers may find it both important and interesting to examine how real economic activities, such as the GDP growth rate, would react to such a generalized intervention and evolve afterwards, in comparison to what would have happened without the intervention. However, the conventional mean-based impulse response method is limited in analyzing such a situation because it only demonstrates how the expected value of the variable of interest changes over the forecast horizon, neglecting the variance or uncertainty of the variable. To accurately assess the impact of an intervention on a variable, understanding how both the expected value and the variance of the variable respond to the intervention is essential. For instance, consider a scenario where an intervention induces a moderate increase (in terms of the expected value) in the GDP growth rate in the future, while simultaneously causing a substantially large increase in the uncertainty level (in terms of the variance) of the GDP growth rate. In such a case, questions may arise regarding the desirability of such an intervention as a relevant policy measure.

To evaluate how the uncertainty level of a variable responds to an external intervention, we propose a new impulse response function called the 'Variance Impulse Response Function (VIRF)' as follows

$$\Delta_V(s) = Var(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) - Var(y_{t+s}|\mathcal{F}_{t-1}), \tag{9}$$

where $s = 0, 1, ..., h$ for some forecast horizon h. The first term computes the conditional variance of the affected observables after the intervention, while the second term measures what could have happened to the same observables if the external intervention had not occurred.

For illustration, let us assume that $\delta = 0$ and $\lambda = diag\{1, \lambda_2\}$. The first term in (9) $Var(\tilde{y}_{t+s}|\mathcal{F}_{t-1})$ can be computed from the MA representation for $\tilde{y}_{t+s}$ in (5) as follows

$$\begin{aligned} Var(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) &= Var(\Phi_0\varepsilon_{t+s} + \Phi_1\varepsilon_{t+s-1} + ... + \Phi_{s-1}\varepsilon_{t+1} + \Phi_s\tilde{\varepsilon}_t|\mathcal{F}_{t-1}) \\ &= \Phi_0\Sigma\Phi_0' + \Phi_1\Sigma\Phi_1' + ... + \Phi_{s-1}\Sigma\Phi_{s-1}' + \Phi_s\Sigma_\lambda\Phi_s', \end{aligned}$$

where $\Sigma_\lambda = \lambda\Sigma\lambda'$. The second term in (9) $Var(y_{t+s}|\mathcal{F}_{t-1})$ is also analogously obtained from (3) as

$$\begin{aligned} Var(y_{t+s}|\mathcal{F}_{t-1}) &= Var(\Phi_0\varepsilon_{t+s} + \Phi_1\varepsilon_{t+s-1} + ... + \Phi_{s-1}\varepsilon_{t+1} + \Phi_s\varepsilon_t|\mathcal{F}_{t-1}) \\ &= \Phi_0\Sigma\Phi_0' + \Phi_1\Sigma\Phi_1' + ... + \Phi_{s-1}\Sigma\Phi_{s-1}' + \Phi_s\Sigma\Phi_s', \end{aligned}$$

which implies that the VIRF is

$$\begin{aligned} \Delta_V(s) &= \Phi_s(\Sigma_\lambda - \Sigma)\Phi_s', \qquad (10) \\ &= (A_0^{-1}A_1)^s A_0^{-1}(\lambda\Sigma\lambda' - \Sigma)A_0^{-1\prime}(A_1'A_0^{-1\prime})^s. \end{aligned}$$

It is also obvious using the result in (7) that the MIRF under such a particular intervention is

$$\Delta_E(s) = 0 \tag{11}$$

for any $s \geq 0$. The result in (11) shows no fluctuations in the mean impulse response function when there is a scale shift-only shock. Such a finding is not surprising because (i) there is no mean shift intervention, and (ii) there is no channel through which the scale shift intervention can affect the conditional mean function of the observables, $y_t = (y_{1t}, y_{2t})'$. A way to open such a channel is to include Garch-in-Mean in the baseline SVAR model.

Before we close this section, we consider the case in which there is a mean shift intervention (i.e., $\delta \neq 0$) without any scale shift intervention (i.e., $\lambda = diag\{1, 1\}$). Then, the formula for the VIRF in (10) implies that

$$\Delta_V(s) = 0 \tag{12}$$

for any $s \geq 0$. This result is not surprising either for the same reason; that is, there is no channel through which the mean-shift-only shock can influence the conditional variance function of y_t. To make such a channel, some heteroscedastic features such as ARCH or GARCH equations can be embedded into our base SVAR model.

To summarize what we have found in plain words, there is no 'crossover effect' in the standard SVAR models without any heteroscedastic features. To clarify the meaning of the term 'crossover effect,' we say that we observe the crossover effect if at least one of the following situations arises: (i) a mean-shift-only intervention results in $\Delta_V(s) \neq 0$ for some values of s, or (ii) a scale-shift-only intervention results in $\Delta_E(s) \neq 0$ for some values of s. In order to have such a crossover effect, we need to include some form of heterogeneity in our model, which will be investigated in detail in the next section.

The generalized intervention scheme in (8) and the concept of uncertainty shocks as a special case when equipped with relevant response functions such as MIRFs or VIRFs can enable researchers to quantify the impact of an uncertainty shock on some variables of interest. For example, we can measure the impact of uncertainty shocks in the oil sector on U.S. aggregate output measured in terms of real GDP growth.

We acknowledge the extensive VAR literature dedicated to modeling uncertainty shocks and assessing their impacts. A prominent approach, inspired by Bloom (2009), involves VAR models incorporating an empirically constructed uncertainty variable. Identification of uncertainty shocks is achieved recursively by appropriately ordering this variable. Subsequently, the traditional mean impulse response function is employed to quantify the impact of the identified uncertainty shock on other variables within the VAR model. Noteworthy papers employing this approach include Fernandez-Villaverde et al. (2015) and Caldara et al. (2020). A similar strand of literature on uncertainty shocks utilizes linear VAR models with innovative identification strategies, such as "shock-based restrictions," diverging from the conventional recursive identification found in mainstream literature. (e.g., Ludvigson et al., 2021). In both approaches, the construction of an empirical uncertainty variable is a necessity. Examples include fiscal volatility shocks to capital tax rates in Fernandez-Villaverde et al. (2015), measures of trade policy uncertainty in Caldara et al. (2020), and macroeconomic uncertainty and financial uncertainty indexes in Ludvigson et al. (2021).

In contrast with these approaches mentioned above, our approach eliminates the need for a separate uncertainty measure. Both relevant uncertainty shocks and level shocks can be automatically generated within the VAR framework upon identification, regardless of the chosen identification strategy whether conventional or otherwise. As a result, our approach offers relative advantages, especially when either constructing an uncertainty measure is challenging or there is controversy over which uncertainty measure to use among multiple alternatives. Since our approach is straightforward to implement within any conventional VAR framework without requiring a specific uncertainty variable, it can serve as a convenient preliminary check for the effects of uncertainty shocks, even in cases where a readily-usable or definite uncertainty measure exists.

There is another significant strand of literature that examines the impact of uncertainty shocks. This relies on the stochastic volatility model within the SVAR framework, as demonstrated by studies such as Mumtaz and Zanetti (2013), Carriero et al. (2018, 2022), and Caldara et al. (2023). This stochastic volatility (SV)-based approach shares similarities with ours in that an external measure of uncertainty is unnecessary. The key distinction lies in how the volatility process is modeled. While the SV-based approach employs a latent volatility process, our approach utilizes the observable GARCH process. These models are non-nested and can be considered as alternative methods.

The SV model assumes two error processes (one for the conditional mean process and another for the latent volatility process), providing potential advantages in in-sample fitting and prediction. However, its likelihood function is intractable for optimization, typically requiring the use of the Markov Chain Monte Carlo (MCMC) method with Gibbs algorithms for estimation. In contrast, GARCH models are easily estimated using standard maximum likelihood methods, making them more accessible for applied researchers.

Our approach offers straightforward and intuitive closed-form formulae for reaction functions to analyze the impact of uncertainty shocks. In contrast, the SV-based approach, as seen in Mumtaz and Zanetti (2013), relies on the usual mean reaction function to evaluate the effects of uncertainty shocks but does not present closed-form solutions for these functions. Additionally, the SV-based approach does not delve into how the variance of the variable of interest responds to external interventions. As previously emphasized, a comprehensive understanding of both the mean and variance responses to an intervention is desirable for an accurate impact assessment. In this paper, we not only provide a closed-form solution for the mean reaction function but also extend this to include the variance reaction function. The subsequent section details these solutions for both the Mean Impulse Response Function (MIRF) and Variance Impulse Response Function (VIRF) under the generalized impulse framework.

5 The Crossover Effect in General Models

In this section, we consider a multi-variate case with y_t being an $n \times 1$ vector given by $y_t = (y_{1t}, y_{2t}, ..., y_{nt})'$. The dimensions of all other variables and coefficients are redefined accordingly. We consider the following full-fledged GARCH-in-Mean SVAR(d)[2] model

$$\begin{aligned} A_0 y_t &= c + A_1 y_{t-1} + ... + A_d y_{t-d} + \Lambda h_t + \varepsilon_t \\ \varepsilon_t &= H_t^{1/2} \eta_t \\ \eta_t &\sim IIN(0, I) \\ Var(\varepsilon_t | \mathcal{F}_{t-1}) &= H_t \end{aligned} \tag{13}$$

where H_t is the conditional variance matrix. For simplicity, we assume that H_t is a diagonal matrix given by $H_t = diag\{h_{1t}, h_{2t}, ..., h_{nt}\}$ with h_{it} following a GARCH(1,1) process[3] as follows:

$$h_{it} = \beta_{0i} + \beta_{1i} \varepsilon_{it-1}^2 + \beta_{2i} h_{it-1}, \tag{14}$$

where β_{0i}, β_{1i}, and β_{2i} are assumed to be restricted to ensure that (i) h_{it} is positive and (ii) the usual stability condition ($\beta_{1i} + \beta_{2i} < 1$) is satisfied. There is another stability condition introduced later, so to distinguish between the two, $\beta_{1i} + \beta_{2i} < 1$ is referred to as the 'mean stability condition.' For more information on the mean stability condition, refer to Bollerslev et al. (1988). The GARCH-in-Mean type model was introduced in the seminal paper by Engle et al. (1987) and has been applied successfully in many empirical studies. When using GARCH-in-Mean models, it is necessary to specify the type of volatility that enters the conditional mean equation. Some popular choices are h_t, $\sqrt{h_t}$, or $\ln(h_t)$. We assume, as in Engle et al. (1987), that the conditional mean equation is dependent on h_t as specified in (13). Our rationale for choosing h_t is based solely on mathematical convenience because this choice will result in mathematically tractable impulse response functions. For example, if we use either $\sqrt{h_t}$ or $\ln(h_t)$ instead of h_t, we need to compute terms such as $E\left(\sqrt{h_t} | \mathcal{F}_{t-1}\right)$ or $E\left(\ln(h_t) | \mathcal{F}_{t-1}\right)$. These terms are difficult to evaluate, and can only be approximated.

For illustration, we start with the simplest form of heterogeneity with $\Lambda = 0$ in (13); that is, (i) the conditional variance function is computed through the GARCH equation in (14), but (ii) the conditional variance does not have any feedback effect on the conditional mean of

[2] We assume that the stationarity condition for the SVAR(d) model is satisfied; that is, all the roots in the equation

$$\det(I - A_1 z - ... - A_d z^d) = 0$$

lie outside the unit circle.

[3] Our framework can be extended to include a GARCH(p, q) process, but for simplicity, we use our simple model.

the model since $\Lambda = 0$. Then, it is straightforward to show under the general impulse scheme in (8) that the MIRF is the same as in the homoscedastic SVAR model in (7). However, the derivation of the VIRF is a little bit challenging, and the result is presented below (the proof is in Appendix A):

$$\Delta_V(s) = \sum_{k=0}^{s-1} \Phi_k diag\{(q_{s-k} - \kappa_{s-k}) \circ (\bar{h}_t - h_t)\}\Phi'_k + \Phi_s(\lambda H_t \lambda' - H_t)\Phi'_s, \tag{15}$$

where $\bar{h}_t = diag\{\bar{h}_{1t}, ..., \bar{h}_{nt}\}$ with $\bar{h}_{it} = \delta_i^2 + \lambda_i^2 h_{it}$ and other notations are defined below. The vector product $a \circ b$ in (15) is defined to multiply corresponding elements in a and b; that is, $a \circ b = (a_1 b_1, ..., a_n b_n)'$. It is observed from (15) that a mean shift intervention alone can generate significant changes in the VIRF as location shift factor δ enters the VIRF formula. Hence, a form of the cross-over effect is now realized in our model.

Under the assumption $\Lambda = 0$, we emphasize that the MIRF remains the same as in the homoscedastic SVAR model, indicating that no scale shift intervention can cause a change in the MIRF. As a result, we see only a limited form of the crossover effect in this case. This is due to the simplifying assumption $\Lambda = 0$. Because of this restriction, no scale shift factor λ can enter the conditional mean equation, and hence, cannot affect the MIRF. In the following, we remove this limitation, and demonstrate that the crossover effect can be fully observed in both directions.

The SVAR(d) process with GARCH-in-Mean ($\Lambda \neq 0$) in (13) can be converted into an MA(∞) process

$$\begin{aligned} y_{t+s} &= \mu + \Phi_0 \varepsilon_{t+s} + \Phi_1 \varepsilon_{t+s-1} + ... + \Phi_{s-1}\varepsilon_{t+1} + \Phi_s \varepsilon_t + \Phi_{s+1}\varepsilon_{t-1} + \Phi_{s+2}\varepsilon_{t-2}.... \\ &\quad + \Psi_0 h_{t+s} + \Psi_1 h_{t+s-1} + ... + \Psi_{s-1} h_{t+1} + \Psi_s h_t + \Psi_{s+1} h_{t-1} + \Psi_{s+2} h_{t-2}... \end{aligned} \tag{16}$$

where $\mu = (I - \sum_{i=1}^{d} A_0^{-1} A_i)^{-1} A_0^{-1} c$, $\Phi_i = \Theta_i A_0^{-1}$, and $\Psi_i = \Theta_i A_0^{-1} \Lambda$. As usual, the MA($\infty$) coefficients Θ_i can be computed from the VAR(d) coefficients as the solutions to the simultaneous difference equations

$$\Theta_k = A_0^{-1}(A_1 \Theta_{k-1} + A_2 \Theta_{k-2} + ... + A_d \Theta_{k-d}) \quad \text{for } k \geq d+1$$

with the following initial conditions

$$\begin{aligned}
\Theta_0 &= I \\
\Theta_1 &= A_0^{-1}(A_1\Theta_0) \\
\Theta_2 &= A_0^{-1}(A_1\Theta_1 + A_2\Theta_0) \\
&\vdots \\
\Theta_d &= A_0^{-1}(A_1\Theta_{d-1} + A_2\Theta_{d-2} + \dots + A_d\Theta_0).
\end{aligned}$$

The MA(∞) process in (16) can be decomposed as follows

$$y_{t+s} = \mu + E_{t,s} + E_{t,s}^C + H_{t,s} + H_{t,s}^C, \tag{17}$$

where

$$\begin{aligned}
E_{t,s} &= \sum_{k=0}^{s} \Phi_k \varepsilon_{t+s-k}, \\
E_{t,s}^C &= \sum_{k=s+1}^{\infty} \Phi_k \varepsilon_{t+s-k}, \\
H_{t,s} &= \sum_{k=0}^{s} \Psi_k h_{t+s-k}, \\
H_{t,s}^C &= \sum_{k=s+1}^{\infty} \Psi_k h_{t+s-k}.
\end{aligned}$$

Such a decomposition is made on the grounds that $E_{t,s}$ and $H_{t,s}$ can be viewed as future values, whereas the other two terms, $E_{t,s}^C$ and $H_{t,s}^C$, can be regarded as past values with respect to the pre-specified information set $\mathcal{F}_{t-1}$. As a result, we have some straightforward results, such as $E(E_{t,s}|\mathcal{F}_{t-1}) = 0$ and $E(E_{t,s}^C|\mathcal{F}_{t-1}) = E_{t,s}^C$. As will be shown, each of the four components in (17) will play a key role in deriving the explicit formulae for both MIRF and VIRF.

Similar to (5), it is necessary to obtain the MA representation for the affected time series $\tilde{y}_{t+s}$. We assume that the generalized impulse, $\tilde{\varepsilon}_t = \delta + \lambda\varepsilon_t$, is applied to the system at time t. Such a shock will affect h_i at time $t+1$, and the affected h_i at time $t+1$, denoted $\tilde{h}_{i,t+1}$, is given by $\tilde{h}_{i,t+1} = \beta_{0i} + \beta_{1i}\tilde{\varepsilon}_{it}^2 + \beta_{2i}h_{it}$. Affecting h_i at time $t+1$ will change $\varepsilon_{i,t+1}$ through the relation $\varepsilon_{i,t+1} = h_{i,t+1}^{1/2}\eta_{t+1}$; i.e., the affected $\varepsilon_{i,t+1}$, denoted $\tilde{\varepsilon}_{i,t+1}$, is $\tilde{\varepsilon}_{i,t+1} = \tilde{h}_{i,t+1}^{1/2}\eta_{t+1}$. This interaction between $\tilde{\varepsilon}_i$ and $\tilde{h}_i$ starting at time t will change all future values of ε_i and

h_i whereas all past values of ε_i and h_i are unchanged.[4]

As a result, we have the following decomposition for the affected time series, $\tilde{y}_{t+s}$:

$$\tilde{y}_{t+s} = \mu + \tilde{E}_{t,s} + E^C_{t,s} + \tilde{H}_{t,s} + H^C_{t,s}, \tag{18}$$

where

$$\begin{aligned}\tilde{E}_{t,s} &= \sum_{k=0}^{s} \Phi_k \tilde{\varepsilon}_{t+s-k}, \\ \tilde{H}_{t,s} &= \sum_{k=0}^{s} \Psi_k \tilde{h}_{t+s-k}.\end{aligned}$$

Using the two decompositions in (17) and (18), it is straightforward to derive the MIRF, which is shown in the following lemma and theorem.

Lemma 1 Under the GARCH-in-Mean SVAR(d) model in (13), we have the following results[5]:
(1) $E(\varepsilon_{i,t+s}|\mathcal{F}_{t-1}) = 0$ for any $s \geq 0$.
(2) $E(\varepsilon^2_{i,t+s}|\mathcal{F}_{t-1}) = E(h_{i,t+s}|\mathcal{F}_{t-1})$ for any $s \geq 0$.
(3) $E(h_{i,t+s}|\mathcal{F}_{t-1}) = w_{i,s}$ for any $s \geq 0$ where $w_{i,s}$ is the solution to the following first-order difference equation

$$\begin{aligned}w_{i,s} &= \beta_{0i} + (\beta_{1i} + \beta_{2i}) w_{i,s-1} \text{ for } s \geq 1, \\ w_{i,0} &= h_{it}.\end{aligned} \tag{19}$$

(4) $E(\tilde{\varepsilon}_{i,t+s}|\mathcal{F}_{t-1}) = \delta_i 1_{[s=0]}$ for any $s \geq 0$.
(5) $E(\tilde{\varepsilon}^2_{i,t+s}|\mathcal{F}_{t-1}) = (\delta_i^2 + \lambda_i^2 h_{it}) 1_{[s=0]} + E(\tilde{h}_{i,t+s}|\mathcal{F}_{t-1}) 1_{[s\geq 1]}$.

[4] We acknowledge there is a subtle issue here. As discussed in the text, all future and past values are naturally defined once the generalized impulse is given at time t. However, it might not be clear what can happen to the conditional volatility of ε_i at time t. One can argue that it should be based on the affected error: that is, $Var(\tilde{\varepsilon}_t|\mathcal{F}_{t-1}) = \lambda^2 h_t$. In this case, the conditional volatility is changed due to the impulse. However, one can equally argue that the conditional volatility at time t depends only on past values through the GARCH specification $h_t = \beta_0 + \beta_1 \varepsilon^2_{t-1} + \beta_{2i} h_{t-1}$. Hence, it is still unchanged in spite of the impulse since all past values, including ε^2_{t-1} and h_{t-1}, are unchanged. In this paper, we adopt the second view. Our justification is as follows. Considering t as a time period such as a month, a quarter, or a year, it is often realistic to assume that system managers or policymakers usually make their decisions at the very end of period t after taking into account all information and all events occurring during time period t. In such a scenario, when the impulse is given at the end of time period t, the conditional volatility at time t cannot be affected by the impulse. Hence, the second view can be justified.

[5] When proving most of the results in Lemma 1, it is quite helpful to use the tower property that $E(E(X|\mathcal{G}_2)|\mathcal{G}_1) = E(X|\mathcal{G}_1)$ for any two sigma fields $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{G}_1 \subset \mathcal{G}_2$. The proof for Lemma 1 is omitted to save space, but is available upon request.

(6) $E(\tilde{h}_{i,t+s}|\mathcal{F}_{t-1}) = \tilde{w}_{i,s}$ for any $s \geq 0$ where $\tilde{w}_{i,s}$ is the solution to the following first-order difference equation

$$\begin{aligned} \tilde{w}_{i,s} &= \beta_{0i} + (\beta_{1i} + \beta_{2i})\tilde{w}_{i,s-1} \text{ for } s \geq 2, \\ \tilde{w}_{i,1} &= \beta_{0i} + \beta_{1i}\bar{h}_{it} + \beta_{2i}h_{it}, \\ \bar{h}_{it} &= \delta_i^2 + \lambda_i^2 h_{it}, \\ \tilde{w}_{i,0} &= h_{it}. \end{aligned} \tag{20}$$

We note that the difference equation in (19) can be recursively solved, and the solution for any $s \geq 1$ is

$$w_{i,s} = p_{i,s} + q_{i,s}h_{it}, \tag{21}$$

where $p_{i,s} = \beta_{0i}\sum_{k=0}^{s-1}(\beta_{1i} + \beta_{2i})^k$ and $q_{i,s} = (\beta_{1i} + \beta_{2i})^s$. Similarly, by solving the difference equation in (20), we can show that its solution is

$$\tilde{w}_{i,s} = p_{i,s} + q_{i,s}\bar{h}_{it} - \kappa_{i,s}(\bar{h}_{it} - h_{it}), \tag{22}$$

where $\kappa_{i,s} = (\beta_{1i} + \beta_{2i})^{s-1}\beta_{2i}$ for $s \geq 1$. The usual stability condition $(\beta_{1i} + \beta_{2i} < 1)$ will ensure that the two solutions are convergent as the forecast horizon s becomes large.

Theorem 1 Under the GARCH-in-Mean SVAR(d) model in (13), the MIRF is given by

$$\begin{aligned} \Delta_E(s) &= \Phi_s\delta + \sum_{k=0}^{s-1}\Psi_k(\tilde{w}_{s-k} - w_{s-k}) \\ &= \Phi_s\delta + \sum_{k=0}^{s-1}\Psi_k\{(q_{s-k} - \kappa_{s-k}) \circ (\bar{h}_t - h_t)\}, \end{aligned}$$

where $\tilde{w}_{s-k} = (\tilde{w}_{1,s-k}, ...\tilde{w}_{n,s-k})'$, $w_{s-k} = (w_{1,s-k}, ...w_{n,s-k})'$, $q_{s-k} = (q_{1,s-k}, ..., q_{n,s-k})'$, and $\kappa_{s-k} = (\kappa_{1,s-k}, ..., \kappa_{n,s-k})'$.

The results in Theorem 1 clearly show that the MIRF is a function not only of mean shift factor δ but also of scale shift factor λ. Hence, a scale shift intervention alone can cause significant changes in the MIRF. Next, we turn to the VIRF, which is derived in Theorem 2 as follows.

Lemma 2 Under the GARCH-in-Mean SVAR(d) model in (13), we have the following results (1) $Var(h_{i,t+s}|\mathcal{F}_{t-1}) = v_{i,s}$ for $s \geq 0$ with $v_{i,s} = x_{i,s} - {w_{i,s}}^2$ where, for $s \geq 1$, $x_{i,s}$ is the solution to the following first-order difference equation:

$$\begin{aligned} x_{i,s} &= a_{i,s} + b_i x_{i,s-1}, \\ a_{i,s} &= \beta_{0i}^2 + 2\beta_{0i}(\beta_{1i} + \beta_{2i})w_{i,s-1}, \\ b_i &= \omega\beta_{1i}^2 + \beta_{2i}^2 + 2\beta_{1i}\beta_{2i}, \\ x_{i,0} &= h_{it}^2, \\ \omega &= E(\eta_t^4). \end{aligned}$$

(2) $Cov(h_{i,t+s}, h_{j,t+s}|\mathcal{F}_{t-1}) = 0$ for $i \neq j$ and $s \geq 0$.
(3) $Cov(h_{i,t+s}, h_{i,t+k}|\mathcal{F}_{t-1}) = \gamma_i(s,k)$ for $1 \leq s < k$ where $\gamma_i(s,k) = p_{i,k-s}w_{i,s} + q_{i,k-s}x_{i,s} - w_{i,s}w_{i,k}$.[6]
(4) $Cov(h_{i,t+s}, h_{j,t+k}|\mathcal{F}_{t-1}) = 0$ for $i \neq j$ and $0 \leq s < k$.
(5) $Var(\tilde{h}_{i,t+s}|\mathcal{F}_{t-1}) = \tilde{v}_{i,s}$ for $s \geq 0$ with $\tilde{v}_{i,s} = \tilde{x}_{i,s} - \tilde{w}_{i,s}^2$ where, for $s \geq 2$, $\tilde{x}_{i,s}$ is the solution to the following first-order difference equation:

$$\begin{aligned} \tilde{x}_{i,s} &= \tilde{a}_{i,s} + b_i\tilde{x}_{i,s-1}, \\ \tilde{a}_{i,s} &= \beta_{0i}^2 + 2\beta_{0i}(\beta_{1i} + \beta_{2i})\tilde{w}_{i,s-1}, \\ \tilde{x}_{i,1} &= \beta_{0i}^2 + 2\beta_{0i}(\beta_{1i}\bar{h}_{it} + \beta_{2i}h_{it}) + 3\beta_{1i}^2(\bar{h}_{it}^2 - 2\delta^4/3) + \beta_{2i}^2h_{it}^2 + 2\beta_{1i}\beta_{2i}h_{it}\bar{h}_{it}, \\ \tilde{x}_{i,0} &= h_{it}^2. \end{aligned}$$

(6) $Cov(\tilde{h}_{i,t+s}, \tilde{h}_{j,t+s}|\mathcal{F}_{t-1}) = 0$ for $i \neq j$ and $s \geq 0$.
(7) $Cov(\tilde{h}_{i,t+s}, \tilde{h}_{i,t+k}|\mathcal{F}_{t-1}) = \tilde{\gamma}_i(s,k)$ for $1 \leq s < k$ where $\tilde{\gamma}_i(s,k) = p_{i,k-s}\tilde{w}_{i,s} + q_{i,k-s}\tilde{x}_{i,s} - \tilde{w}_{i,s}\tilde{w}_{i,k}$.
(8) $Cov(\tilde{h}_{i,t+s}, \tilde{h}_{j,t+k}|\mathcal{F}_{t-1}) = 0$ for $i \neq j$ and $0 \leq s < k$.
(9) $Cov(\varepsilon_{i,t+s}, h_{j,t+k}|\mathcal{F}_{t-1}) = 0$ for all i,j and all $s \geq 0, k \geq 0$.
(10) $Cov(\tilde{\varepsilon}_{i,t+s}, \tilde{h}_{j,t+k}|\mathcal{F}_{t-1}) = 0$ for all i,j and all $s \geq 0, k \geq 0$.

We note that $E(\eta_t^4) = 3$ under the normality assumption imposed in (13). Hence, we will use the normal value $\omega = 3$ when computing conditional variance term $Var(h_{i,t+s}|\mathcal{F}_{t-1})$ appearing in Lemma 2(1) in our numerical analysis as well as for empirical analysis in the following sections. The normality assumption is also used in other places in proving Lemma 2. For example, to prove Lemma 2(2), we need to use $E(\eta_{it}^2\eta_{jt}^2) = E(\eta_{it}^2)E(\eta_{jt}^2)$, which is justified by the joint normality assumption η_t. Another point worth mentioning is that

[6] We note that $\gamma_i(0,k) = 0$.

initial value $\tilde{x}_{i,1}$ seems quite involved, but it can be even more complicated in the absence of the symmetry about zero in the distribution of η_t, which is assumed in our model since η_t is normal with mean zero. This is because terms such as $E(\eta_{it}^3)$ appear in the derivation of $\tilde{x}_{i,1}$. We also note that conditional variance term $Var(h_{i,t+s}|\mathcal{F}_{t-1})$ in Lemma 2(1) is expressed in terms of $x_{i,s}$, which is in the form of a difference equation of order 1 with its coefficient $b_i = \omega\beta_{1i}^2 + \beta_{2i}^2 + 2\beta_{1i}\beta_{2i}$. Hence, to ensure the stability of conditional variance term $Var(h_{i,t+s}|\mathcal{F}_{t-1})$, we will assume that $\omega\beta_{1i}^2 + \beta_{2i}^2 + 2\beta_{1i}\beta_{2i} < 1$, which will be referred to as the 'variance stability condition.' For a detailed discussion on the variance stability condition, please see Appendix B.

Theorem 2 Under the GARCH-in-Mean SVAR(d) model in (13), the VIRF is given by

$$\Delta_V(s) = \Delta_1(s) + \Delta_2(s),$$

where

$$\Delta_1(s) = \sum_{k=0}^{s-1} \Phi_k diag\{(q_{s-k} - \kappa_{s-k}) \circ (\bar{h}_t - h_t)\}\Phi_k' + \Phi_s(\lambda H_t \lambda' - H_t)\Phi_s',$$

and

$$\begin{aligned}\Delta_2(s) &= \sum_{k=0}^{s} \Psi_k diag\{\tilde{v}_{s-k} - v_{s-k}\}\Psi_k' \\ &\quad + 2\sum_{k=1}^{s}\sum_{j=0}^{k-1} \Psi_k diag\{\tilde{\gamma}(s-k, s-j) - \gamma(s-k, s-j)\}\Psi_j',\end{aligned}$$

with

$$\begin{aligned} v_{s-k} &= (v_{1,s-k}, ..., v_{n,s-k})', \\ \tilde{v}_{s-k} &= (\tilde{v}_{1,s-k}, ..., \tilde{v}_{n,s-k})', \\ \gamma(s-k, s-j) &= (\gamma_1(s-k, s-j), ..., \gamma_n(s-k, s-j))', \\ \tilde{\gamma}(s-k, s-j) &= (\tilde{\gamma}_1(s-k, s-j), ..., \tilde{\gamma}_n(s-k, s-j))'. \end{aligned}$$

Theorems 1 & 2 demonstrate that both the MIRF and the VIRF are functions of all relevant population parameters as well as conditional variance term h_{it} due to the inclusion of h_{it} in conditioning information set $\mathcal{F}_{t-1}$. When estimating these impulse response functions, all the relevant parameters can be replaced by their ML estimators, but we need to decide which method to use to evaluate h_{it}. In the following numerical and empirical analyses, we replace h_{it} with its expected value, $E(h_{it})$, which is given by $E(h_{it}) = \beta_{0i}(1 - \beta_{1i} - \beta_{2i})^{-1}$.

Recalling the definition of VIRF as $\Delta_V(s) = Var(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) - Var(y_{t+s}|\mathcal{F}_{t-1})$, the main focus is on the diagonal elements of $n \times n$ matrix $\Delta_V(s)$, because they reveal the impact of an intervention on the conditional variances of the variables of interest. The expression of the off-diagonal elements of $\Delta_V(s)$, denoted $\Delta_{C(i,j)}(s)$, is given by $\Delta_{C(i,j)}(s) = Cov(\tilde{y}_{i,t+s}, \tilde{y}_{j,t+s}|\mathcal{F}_{t-1}) - Cov(y_{i,t+s}, y_{j,t+s}|\mathcal{F}_{t-1})$ for $i \neq j$, and this function is referred to as the 'Covariance Impulse Response Function (CIRF).' Exploring the CIRF can be highly informative, because it sheds light on the manner in which conditional covariance functions react to a specific intervention.

It is worth noting that the acronym VIRF has already been used in the literature. Hafner and Herwartz (2006) introduced a novel notion of impulse response function, referred to as the 'Volatility Impulse Response Function (VIRF).' Their impulse response function can be expressed in terms of our notations

$$VIRF = E(\tilde{h}_{t+s}|\mathcal{F}_{t-1}) - E(h_{t+s}|\mathcal{F}_{t-1}).$$

It is evident that the volatility impulse response function in Hafner and Herwartz's (2006) measures the response of the expected value of the conditional variance of shock ε_{t+s} to a mean shift intervention, which is different from our VIRF. In our approach, we are primarily interested in how the conditional variance of the endogenous variables, y_{t+s}, responds to both mean shift interventions and scale shift interventions.

In the previously discussed SV-based approach, stochastic volatility processes are specified for h_{it} instead of the GARCH equation (14). For instance, in Mumtaz and Zanetti (2013), the variance matrix H_t in equation (13) assumes a diagonal structure similar to our setup. However, the diagonal elements are given by $\exp(h_{it})$ where the log-volatility h_{it} follows an AR(1) process $h_{it} = \theta_i h_{it-1} + \eta_{it}$ with $\eta_{it} \sim N(0, Q_{ii})$. Therefore, a conventional mean shift intervention to η_{it} can induce uncertainty shocks within the SV-based approach, and the response of endogenous variables y_t to such uncertainty shocks can be measured using the conventional MIRF. As previously mentioned, however, the SV-based approach does not offer closed-form solutions for their MIRF and does not consider any variance reaction function such as our VIRF in Theorem 2.

6 Numerical Analysis

The proposed impulse response functions in Theorems 1 and 2 are too complicated to be interpreted intuitively. Even the MIRF loses its simple and appealing interpretability of the impact of an intervention as in the standard literature. As shown in Theorem 1, the first term retains such an implication, but the second term is difficult to interpret, let alone the VIRF formula shown in Theorem 2. Hence, we numerically investigate how these response functions behave under various settings of model parameters and intervention schemes.

The DGP used in our numerical analysis is the GARCH-in-Mean SVAR(d) model model in (13) with two endogenous variables ($n = 2$) only. We start with the simplest setting with model parameters, specified as follows:

$$\begin{aligned}
A_0 &= \begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix}, \\
c &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0.6 & -0.3 \\ -0.1 & 0.6 \end{bmatrix}, \\
\Lambda &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
\beta_{0i} &= 1, \beta_{1i} = 0.4, \beta_{2i} = 0.4, \\
\varepsilon_{it} &= h_{it}^{1/2}\eta_{it}, \\
\eta_{it} &\sim IIN(0, 1).
\end{aligned}$$

Hence, the first model we use is the SVAR($d = 1$) model in which there is no GARCH-in-Mean effect. A mean shift intervention is given only to the structural shock of y_{2t}, and the size of intervention is one standard deviation of the error term (ε_{2t}); that is, $\sigma_2 = \sqrt{var(\varepsilon_{2t})}$ which is about 2.24. Therefore, the intervention matrices are formally specified as

$$\begin{aligned}
\delta &= \begin{bmatrix} 0 \\ \sigma_2 \end{bmatrix}, \qquad (23) \\
\lambda &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Since λ is the identity matrix, there is no uncertainty shock to variables y_{1t} and y_{2t}. This is a typical specification, and the resulting MIRFs of variables y_{1t} and y_{2t} are shown in Figure 1.[7] The MIRF of y_{1t} is indicated by the solid line, whereas the MIRF of y_{2t} is the dashed line.

[7] The formulae for the MIRF and the VIRF shown in Theorems 1 & 2 are programmed in MATLAB. All MATLAB codes and the data set used in the paper can be obtained from the corresponding author upon request.

Since the intervention is made to increase the location of the shock of y_{2t} by one standard deviation of ε_{2t}, the MIRF of y_{2t} starts at that value and begins to decrease. The impact of such an intervention dies out after about 35 periods. The MIRF of y_{2t} starts to decrease, but soon it begins to increase.

Next we examine what can happen to the VIRF when there is a mean-shift-only intervention. Even though the intervention is only to the location of y_{2t}, the VIRF responds to such an intervention significantly, and the resulting VIRFs are shown in Figure 2. Both VIRFs of y_{1t} and y_{2t} initially begin to increase from zero, and soon start to decrease, although y_{2t} reacts more aggressively to the mean shift intervention than y_{1t}. Hence, given our model specification, a positive location shift intervention on ε_{2t} is likely to decrease the mean of y_{1t} as expected, but more interestingly, the dispersion of both variables becomes larger after such an intervention.

One might ask why a mean shift impulse can generate significant changes not only in the mean impulse response function, but also in the variance impulse response function. This kind of phenomenon has been observed in Lee et al. (2021) using real macroeconomic data. By employing quantile impulse response functions (QIRFs) proposed in their paper, they show that the dispersion of the employment growth distribution changes after an unexpected monetary contraction (a location shift intervention) in their SVAR model. In the absence of the VIRF method, they indirectly measure the size of the dispersion based on quantile distance, such as the interquartile range, using their quantile impulse response functions. Such a phenomenon is interesting, but Lee et al (2021) does not explain why it happens. To answer this, we attempt to compute the same VIRF under the same setting as in Figure 2, but by switching off the GARCH equation; i.e., letting $\beta_{1i} = 0$ and $\beta_{2i} = 0$. In this case, the VIRF is a flat function at zero. Hence, it can be argued that the main drive for generating significant changes in either the QIRF or the VIRF might be some heteroscedastic structure embedded in the system.

As mentioned in Section 5, we can also compute the CIRF $\Delta_{C(1,2)}(s) = Cov(\tilde{y}_{1,t+s}, \tilde{y}_{2,t+s}|\mathcal{F}_{t-1}) - Cov(y_{1,t+s}, y_{2,t+s}|\mathcal{F}_{t-1})$ shown in Theorem 2. The graph of the CIRF is displayed in Figure 3. The figure shows that the covariance between y_{1t} and y_{2t} becomes negative after a location shift intervention on y_{2t}. When we change the sign of the intervention (i.e., $\delta_2 = -\sigma_2$) in the benchmark specification, the MIRFs, as expected, are flipped over, maintaining their identical shapes, whereas the VIRFs as well as the CIRF are unchanged. We do not report the figures to save space.

We now attempt to change how we intervene in the system. The specification is exactly the same as in the benchmark case, but now there is a scale shift intervention only; that is, $\delta_1 = \delta_2 = 0$, $\lambda_1 = 1$, and $\lambda_2 = 1.5$. Therefore, the level of uncertainty (standard deviation) of the shock has been unexpectedly increased by 50%. When such an intervention occurs, the MIRF does not respond because there is still no feedback from the variance equation to the mean equation in the benchmark specification; i.e., there is no GARCH-in-Mean effect yet. Hence, the MIRFs are all flat at zero. However, the VIRFs as well as the CIRF respond as shown in Figure 4. Since the intervention is to increase the scale of the shock of y_{2t} by 50%, the VIRF of y_{2t} starts immediately at a positive value (about 6) and then begins to rapidly decrease. The VIRF of y_{1t} starts to increase, reaching its maximum value at about 1.5, and then starts to decrease. Note in the figure that the unexpected increase in the level of uncertainty makes the CIRF between y_{1t} and y_{2t} negative.

Finally, we attempt to inject the GARCH-in-Mean effect into the model by setting

$$\Lambda = \begin{bmatrix} -0.4 & 0 \\ 0 & -0.4 \end{bmatrix},$$

which means that a higher variance is likely to result in a lower expected value in the data generating process. We start with the intervention given only at the location of the structural shock to y_{2t}, and the size of the intervention is the same as in Figure 1. The resulting MIRFs are shown in Figure 5. Compared with Figure 1, the impact of the same intervention on y_{1t} is much larger in terms of both magnitude and persistence. The MIRF of y_{1t} decreases more negatively and returns to its steady state more slowly. The only difference between Figure 1 and Figure 5 is the presence of the GARCH-in-Mean effect, but the intervention scheme is identical in both figures. Hence, we can conjecture that if the GARCH-in-Mean effect is mistakenly omitted in modeling, then the MIRF can be misleading, and could have been greatly under-estimated. All the VIRFs increase as shown in Figure 6. Similar to the MIRF, they respond more aggressively and more persistently when compared to their counterparts in Figure 2. However, the level of magnitude and persistence displayed in the VIRFs is somewhat spectacular. The maximum values, compared to the corresponding values in Figure 2, are much larger for both y_{1t} and y_{2t}, respectively. Moreover, such large effects diminish slowly, so they are far from zero even at forecast horizon $h = 35$, which is in sharp contrast with the cases in Figure 2 where all the effects die out completely at that forecast horizon. The same kind of observations can be seen for the CIRF between y_{1t} and y_{2t}, which is also presented in Figure 6.

We now switch off the location shift intervention, and turn on the scale-shift-only intervention by setting $\delta_1 = \delta_2 = 0$, $\lambda_1 = 1$, and $\lambda_2 = 1.5$, which are identical to the intervention used in Figure 4. The resulting MIRFs are presented in Figure 7. We wish to emphasize there is no location shift intervention. Even in the absence of a location shift intervention, the reactions of the MIRFs are surprisingly noticeable. The MIRF of y_{1t} reaches its minimum value (-1.7) at forecast horizon $h = 4$, and then increases slowly. The behavior of the MIRF of y_{1t} in terms of magnitude and persistence is similar to Figure, 5 where a location-shift-only intervention is applied. Under the current model parameter setting, two alternative policies (i.e., one to unexpectedly increase the level of y_{2t}, and the other to unexpectedly increase the uncertainty of y_{2t}) seem to achieve the same results. The behaviors of the VIRFs and the CIRF are shown in Figure 8. Similar to the MIRF case, their shapes are comparable to those in Figure 6, but the size of the impact is greater.

7 Empirical Application

In this section, we apply our generalized impulse framework to investigate the impact of oil shocks on U.S. aggregate output measured in terms of real GDP growth. There exist many different specifications about how to model the relation between these two variables. Following Elder and Serletis (2010), we employ a bivariate SVAR(d) model with the GARCH-in-Mean effect where quarterly observations on real oil prices and real GDP growth are used. For comparison, we use exactly the same data set as in Elder and Serletis (2010), but extend the sample period to include more recent observations. Specifically, for the price of oil, we use quarterly observations of US crude oil composite acquisition cost by refiners (RAC) expressed in terms of US dollars per barrel, which are downloadable from the U.S. Energy Information Administration data archive. This index is a weighted average of domestic and imported crude oil costs, and the costs include all charges associated with the acquisition, transportation, and storage of crude oil paid by refiners. In line with Elder and Serletis (2010), we deflate the RAC by the GDP deflator to obtain the real RAC (RRAC). For aggregate output, we use quarterly observations of real GDP (RGDP) expressed in 2012 dollars and seasonally adjusted. All data on both the GDP deflator and the RGDP can be obtained from the data archive of the Federal Reserve Bank of St. Louis.

We then take log differences to produce growth rates of both variables as follows

$$\begin{aligned} y_{1t} &= 400(\ln(RRAC_t) - \ln(RRAC_{t-1})), \\ y_{2t} &= 400(\ln(RGDP_t) - \ln(RGDP_{t-1})), \end{aligned}$$

where each log difference is multiplied by 400 to annualize growth rates expressed as percentage. These two growth rates are used in the bivariate GARCH-in-Mean SVAR(d) model as specified in (13). The sample period of Elder and Serletis (2010) on these growth rates is from 1974:Q2 to 2008:Q1, but we extend the sample period up to 2019:Q4 just prior to the Covid-19 pandemic, which results in 183 quarterly observations for use in estimating the model. The time-series plots of the two variables (y_{1t}, y_{2t}) are shown in Figure 9 and their summary statistics are provided in Table 1. As in Elder and Serletis (2010), we employ four lags in our empirical analysis. Therefore, the following SVAR model is estimated

$$\begin{aligned}
A_0 y_t &= c + A_1 y_{t-1} + A_2 y_{t-2} + A_3 y_{t-3} + A_4 y_{t-4} + \Lambda h_t + \varepsilon_t, \qquad (24)\\
\varepsilon_t &= H_t^{1/2} \eta_t,\\
\eta_t &\sim IIN(0, I),\\
Var(\varepsilon_t | \mathcal{F}_{t-1}) &= H_t = \begin{bmatrix} h_{1t} & 0 \\ 0 & h_{2t} \end{bmatrix},\\
h_{it} &= \beta_{0i} + \beta_{1i}\varepsilon_{it-1}^2 + \beta_{2i} h_{it-1}.
\end{aligned}$$

We impose the usual Cholesky identification assumption restricting contemporaneous coefficient matrix A_0 to a lower triangular matrix. Given the ordering of the two variables that the first is the growth rate of real oil prices and the second is the growth rate of real GDP, the identification assumption implies that real GDP growth can respond to contemporaneous changes in the growth rate of oil prices, but cannot have any contemporaneous impact on oil prices.

The estimation results for the GARCH(1,1) equations in (24) and their stability conditions are shown in Tables 2 and 3. The size and significance of the coefficients of the GARCH equations are in line with the results in Elder and Serletis (2010) except that the coefficient of the GARCH term in the oil equation is significant here, whereas it is not in Elder and Serletis (2010). Even though the variance-stability condition implies the mean-stability condition, we show them separately in Table 3. It is shown that the variance-stability condition seems to be almost binding for the oil GARCH equation while the GDP GARCH equation safely satisfies the condition. The mean-stability condition is satisfied for both equations.

Let us now shift our focus to the coefficients within the mean equation in (24). Specifically, our attention is directed towards contemporaneous coefficient matrix A_0, which comprises only one parameter. This parameter, denoted by the (2,1) element of A_0, captures the degree to which changes in the real price of oil affect real GDP growth contemporaneously. Through estimation, we find that the estimated value of this element is -0.0057, accompanied by a t-statistic of -1.8104. The parameter exhibits marginal significance at the 10% level. This indicates that the real price of oil exerts a positive contemporaneous influence on the GDP growth rate. This result is in line with the positive correlation observed in Table 1.

Among the lagged terms in the mean equation in (24), we show only the results for coefficient matrix A_1 associated with the first lagged term in order to save space. The estimation results for this matrix are presented in Table 4. As expected, each variable demonstrates a positive autocorrelation with its own first lag. Turning to the off-diagonal terms of A_1, we observe that the first lag of the GDP growth rate does not exert a significant impact on oil prices. In contrast, the first lag of the oil price exhibits a negative effect on GDP growth. This suggests that while an increase in oil prices positively influences the contemporaneous GDP growth rate, it has a detrimental effect on future GDP growth.

The impact of the GARCH-in-Mean terms h_{1t} and h_{2t}, indicated by Λ, in equation (24) is examined in Table 5. Interestingly, these terms have no effect on the real price of oil. Furthermore, the GDP growth rate is unaffected by its own volatility. However, the volatility of oil demonstrates a significant negative effect on GDP growth. The estimated coefficient is -0.0002, with a t-statistic of -2.234. Although the magnitude of this effect may seem small, its true consequences will be revealed through the VIRFs developed in this paper, which will be discussed in the subsequent analysis.

We proceed to analyze the results obtained from impulse response analysis using our generalized framework. In the baseline scenario, we consider a location shift impulse applied to the real price of oil without any uncertainty shock. In this case, the magnitude of the intervention is set equal to the standard deviation of the structural error (ε_{1t}) in the oil price equation. Based on the corresponding residuals, the estimated value of this standard deviation is 53.24%. This value is comparable to the annualized standard deviation (55.89%) of oil prices presented in Table 1.[8] Hence, we set $\delta_1 = 53.24$, $\delta_2 = 0$, $\lambda_1 = 1$, and $\lambda_2 = 1$.

The response of the GDP growth rate to this level shift intervention is illustrated in Figure 10. The figure shows the MIRF of the GDP growth rate accompanied by one-standard deviation confidence bands constructed using a bootstrap method. We employ the wild bootstrap method studied by Wu (1986), Liu (1988), and Davidson & Flachaire (2008), with 1,000 bootstrap samples used for computing the confidence bands. The MIRF reveals that following an unexpected increase in oil prices, the GDP growth rate experiences a decline that persists until the fourth quarter, resulting in a downward revision of approximately 60 basis points in terms of annualized GDP growth. Subsequently, the MIRF begins to recover but remains negative until around the 14th quarter, after which it loses statistical significance. These findings indicate that an unexpected increase (of one-standard deviation in magnitude) in oil prices is likely to exert a negative effect on GDP growth over an extended period. It is worth noting that the behavior of the MIRF of the GDP growth rate in response to a positive oil shock closely resembles the corresponding impulse response function discussed in Elder and Serletis (2010).

[8] The impulse response functions reported in Elder and Serletis (2010) are derived from an oil shock with a magnitude equal to the annualized unconditional standard deviation of the change in the real price of oil, which amounts to 56.79%. The corresponding value obtained from our extended dataset aligns closely with this at 55.89%, as shown in Table 1.

Figure 11 showcases the VIRF of the GDP growth rate in response to the location-shift-only impulse. The figure demonstrates that the uncertainty of output, measured in terms of variance, experiences a significant increase until approximately the 12th quarter (3 years), after which it gradually begins to decrease. Its persistence is especially noteworthy. Even after 40 quarters (10 years), the level of uncertainty has only reduced by less than 10% compared to its maximum value, demonstrating a slow decline. Additional results not presented here indicate that even after 100 quarters, the level of uncertainty remains at 0.55 (reduced by approximately half). Consequently, the impact of an unexpected increase in oil prices on the associated uncertainty appears to have a near-permanent effect. We conjecture that this phenomenon is likely attributed to the nearly binding variance-stability condition observed in the oil GARCH equation, as shown in Table 3.

An interesting question to explore is how the covariance between GDP and oil prices responds to such an oil shock. Our proposed CIRF, depicted in Figure 12, can shed light on this matter. Despite the positive contemporaneous correlation between GDP and oil prices presented in Table 1, the CIRF reveals that in the presence of a positive oil shock, the correlation transitions to a negative state and is likely to persist for an extended period, approximately 40 quarters (10 years).

Up to this point, we have focused on examining response functions resulting from a mean shift impulse applied to oil prices. Now, we turn our attention to a different scenario in which there is no mean shift, but an uncertainty shift impulse is introduced to oil prices. In other words, there is no unexpected increase in oil prices, but there is an unexpected increase in the level of uncertainty surrounding oil prices. We assume that the uncertainty level of oil prices increases by 50%, setting $\delta_1 = 0$, $\delta_2 = 0$, $\lambda_1 = 1.5$ and $\lambda_2 = 1$. The reactions of oil prices and GDP growth to this uncertainty shock are depicted in Figures 13 and 14, respectively. When the level of uncertainty in oil prices is unexpectedly heightened, Figure 13 illustrates that oil prices exhibit a substantial and persistent increase. This finding suggests that oil producers may not favor a high degree of predictability regarding future oil prices. Conversely, as shown in Figure 14, the GDP growth rate is adversely impacted by this kind of intervention. It experiences a decline of up to -0.4% in the third quarter, and this negative effect persists even after five years.

Figure 15 displays the VIRF of the GDP growth rate in response to the uncertainty shock. The figure reveals that the uncertainty level of output experiences a sharp increase until approximately the 12th quarter (3 years). The maximum value observed is approximately 1.2. Once again, the persistence of the VIRF is notable because it declines at a very slow pace. Finally, Figure 16 presents the CIRF between the GDP growth rate and oil prices. Similar to the mean shift impulse case, the correlation between GDP and oil prices undergoes a transition to a negative state and remains in that state for an extended period.

8 Conclusion

This paper introduces the concept of 'generalized impulse.' This concept refers to a one-time external intervention applied to a system designed to manipulate the distribution of internal structural errors within the system. The standard mean shift impulse is a special case of the generalized impulse and uncertainty shocks emerge as a special case within the framework of generalized impulses.

In addition, we propose novel impulse response functions, namely the 'Variance Impulse Response Function' and the 'Covariance Impulse Response Function.' These functions provide researchers with a valuable tool to quantify the effects of uncertainty shocks and examine their impacts systematically. By employing these analytical tools, we contribute to a deeper understanding of the consequences and implications of uncertainty shocks in economic systems.

Our newly developed methods are applied to examine the effects of an uncertainty shock in oil prices on the GDP growth rate by utilizing data from the United States. The findings reveal that when the level of uncertainty in oil prices unexpectedly increases, there is a notable and enduring rise in oil prices. Conversely, GDP growth is adversely affected by this heightened uncertainty. The negative impact induced by the increased uncertainty shock of oil prices persists even after a five-year period. Consequently, our results suggest that an unforeseen rise in the uncertainty level of oil prices can have effects similar to those caused by an unexpected increase in oil prices themselves. However, the detrimental effects of uncertainty shocks on the GDP growth rate tend to be more enduring and persistent compared to level shocks.

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Appendix A

Proof of Theorem 1. Given the two MA(∞) representations in (17) and (18), we have

$$E(y_{t+s}|\mathcal{F}_{t-1}) = \mu + E(E_{t,s}|\mathcal{F}_{t-1}) + E(E_{t,s}^C|\mathcal{F}_{t-1}) + E(H_{t,s}|\mathcal{F}_{t-1}) + E(H_{t,s}^C|\mathcal{F}_{t-1}),$$

and

$$E(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) = \mu + E(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) + E(E^C_{t,s}|\mathcal{F}_{t-1}) + E(\tilde{H}_{t,s}|\mathcal{F}_{t-1}) + E(H^C_{t,s}|\mathcal{F}_{t-1}).$$

Therefore, the MIRF is given by

$$\begin{aligned}\Delta_E(s) &= E(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) - E(y_{t+s}|\mathcal{F}_{t-1}) \\ &= \left\{E(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) - E(E_{t,s}|\mathcal{F}_{t-1})\right\} \\ &\quad + \left\{E(\tilde{H}_{t,s}|\mathcal{F}_{t-1}) - E(H_{t,s}|\mathcal{F}_{t-1})\right\}.\end{aligned}$$

The following results are directly derived from Lemma 1. First, we have by Lemma 1(1)

$$E(E_{t,s}|\mathcal{F}_{t-1}) = 0,$$

and by Lemma 1(2) and Lemma 1(3):

$$E(H_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s-1} \Psi_k w_{s-k} + \Psi_s h_t,$$

where $w_{s-k} = (w_{1,s-k}, ...w_{n,s-k})'$. Lemma 1(4) justifies the following:

$$E(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) = \Phi_s \delta,$$

and finally, Lemma 1(5) and Lemma 1(6) are used to show

$$E(\tilde{H}_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s-1} \Psi_k \tilde{w}_{s-k} + \Psi_s h_t,$$

where $\tilde{w}_{s-k} = (\tilde{w}_{1,s-k}, ...\tilde{w}_{n,s-k})'$. Collecting all the terms and using the solutions in (21) and (22), we have the desired result:

$$\begin{aligned}\Delta_E(s) &= \Phi_s \delta + \sum_{k=0}^{s-1} \Psi_k(\tilde{w}_{s-k} - w_{s-k}) \\ &= \Phi_s \delta + \sum_{k=0}^{s-1} \Psi_k\{(q_{s-k} - \kappa_{s-k}) \circ (\bar{h}_t - h_t)\}.\end{aligned}$$

The proof is complete. ■

Proof of Theorem 2. Given the two MA(∞) representations in (17) and (18), we have

$$Var(y_{t+s}|\mathcal{F}_{t-1}) = Var(E_{t,s}|\mathcal{F}_{t-1}) + Var(H_{t,s}|\mathcal{F}_{t-1}) + 2Cov(E_{t,s}, H_{t,s}|\mathcal{F}_{t-1}),$$

and similarly,

$$Var(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) = Var(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) + Var(\tilde{H}_{t,s}|\mathcal{F}_{t-1}) + 2Cov(\tilde{E}_{t,s}, \tilde{H}_{t,s}|\mathcal{F}_{t-1}).$$

Therefore, the VIRF is given by

$$\begin{aligned} \Delta_V(s) &= Var(\tilde{y}_{t+s}|\mathcal{F}_{t-1}) - Var(y_{t+s}|\mathcal{F}_{t-1}) \qquad (25) \\ &= \left\{Var(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) - Var(E_{t,s}|\mathcal{F}_{t-1})\right\} \\ &\quad + \left\{Var(\tilde{H}_{t,s}|\mathcal{F}_{t-1}) - Var(H_{t,s}|\mathcal{F}_{t-1})\right\} \\ &\quad +2\left\{Cov(\tilde{E}_{t,s}, \tilde{H}_{t,s}|\mathcal{F}_{t-1}) - Cov(E_{t,s}, H_{t,s}|\mathcal{F}_{t-1})\right\} \end{aligned}$$

Among the three terms in (25), the first term, $Var(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) - Var(E_{t,s}|\mathcal{F}_{t-1})$, is given in (15) in the text; that is,

$$\Delta_1(s) = \sum_{k=0}^{s-1} \Phi_k diag\{(q_{s-k} - \kappa_{s-k}) \circ (\bar{h}_t - h_t)\}\Phi_k' + \Phi_s(\lambda H_t \lambda' - H_t)_t \Phi_s'. \qquad (26)$$

Now, we prove this result. First, consider

$$Var(E_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s} \Phi_k Var(\varepsilon_{t+s-k}|\mathcal{F}_{t-1})\Phi_k' + 2\sum_{k=1}^{s}\sum_{j=0}^{k-1} \Phi_k Cov(\varepsilon_{t+s-k}, \varepsilon_{t+s-j}|\mathcal{F}_{t-1})\Phi_j'. \quad (27)$$

Note that

$$Var(\varepsilon_{i,t+s-k}|\mathcal{F}_{t-1}) = w_{i,s-k},$$

by Lemma 1(2) and 1(3), and we can easily show that $Cov(\varepsilon_{i,t+s-k}, \varepsilon_{j,t+s-k}|\mathcal{F}_{t-1}) = 0$ for all $i \neq j$, which implies that

$$Var(\varepsilon_{t+s-k}|\mathcal{F}_{t-1}) = diag\{w_{s-k}\}.$$

Next, we can show that $Cov(\varepsilon_{t+s-k}, \varepsilon_{t+s-j}|\mathcal{F}_{t-1}) = 0$ for $j < k$, which implies that the second term in (27) is zero. Hence, we have

$$\begin{aligned} Var(E_{t,s}|\mathcal{F}_{t-1}) &= \sum_{k=0}^{s} \Phi_k diag\{w_{s-k}\}\Phi_k' \\ &= \sum_{k=0}^{s-1} \Phi_k diag\{w_{s-k}\}\Phi_k' + \Phi_s diag\{h_t\}\Phi_s'. \end{aligned}$$

Next, we consider

$$Var(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s} \Phi_k Var(\tilde{\varepsilon}_{t+s-k}|\mathcal{F}_{t-1})\Phi_k' + 2\sum_{k=1}^{s}\sum_{j=0}^{k-1} \Phi_k Cov(\tilde{\varepsilon}_{t+s-k}, \tilde{\varepsilon}_{t+s-j}|\mathcal{F}_{t-1})\Phi_k'.$$

First, we know from Lemma 1(4), 1(5), and 1(6) that

$$Var(\tilde{\varepsilon}_{i,t+s-k}|\mathcal{F}_{t-1}) = \tilde{w}_{i,s-k}1_{[k<s]} + \lambda_i^2 h_{it}1_{[k=s]},$$

and we can show that $Cov(\tilde{\varepsilon}_{i,t+s-k}, \tilde{\varepsilon}_{j,t+s-k}|\mathcal{F}_{t-1}) = 0$ for all $i \neq j$. Hence, we have

$$Var(\tilde{\varepsilon}_{t+s-k}|\mathcal{F}_{t-1}) = diag\{\tilde{w}_{s-k}\}1_{[k<s]} + \lambda H_t \lambda' 1_{[k=s]}.$$

We can also show that $Cov(\tilde{\varepsilon}_{t+s-k}, \tilde{\varepsilon}_{t+s-j}|\mathcal{F}_{t-1}) = 0$ or $j < k$. Therefore, we have

$$Var(\tilde{E}_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s-1} \Phi_k diag\{\tilde{w}_{s-k}\}\Phi_k' + \Phi_s(\lambda H_t \lambda')_t \Phi_s'.$$

Hence, we have

$$\begin{aligned} \Delta_1(s) &= \sum_{k=0}^{s-1} \Phi_k diag\{\tilde{w}_{s-k} - w_{s-k}\}\Phi_k' + \Phi_s(\lambda H_t\lambda' - H_t)_t\Phi_s' \\ &= \sum_{k=0}^{s-1} \Phi_k diag\{(q_{s-k} - \kappa_{s-k}) \circ (\bar{h}_t - h_t)\}\Phi_k' + \Phi_s(\lambda H_t\lambda' - H_t)_t\Phi_s'. \end{aligned} \tag{28}$$

To obtain second term $\Delta_2(s) = Var(\tilde{H}_{t,s}|\mathcal{F}_{t-1}) - Var(H_{t,s}|\mathcal{F}_{t-1})$, we first calculate $Var(H_{t,s}|\mathcal{F}_{t-1})$ as follows:

$$Var(H_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s} \Psi_k Var(h_{t+s-k}|\mathcal{F}_{t-1})\Psi_k' + 2\sum_{k=1}^{s}\sum_{j=0}^{k-1} \Psi_k Cov(h_{t+s-k}, h_{t+s-j}|\mathcal{F}_{t-1})\Psi_j'.$$

Lemma 2(1) and 2(2) show that

$$Var(h_{t+s-k}|\mathcal{F}_{t-1}) = diag\{v_{s-k}\}.$$

Lemma 2(3) and 2(4) also deliver that for $j < k$,

$$Cov(h_{t+s-k}, h_{t+s-j}|\mathcal{F}_{t-1}) = diag\{\gamma(s-k, s-j)\}.$$

Hence, we have

$$Var(H_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s} \Psi_k diag\{v_{s-k}\}\Psi_k' + 2\sum_{k=1}^{s}\sum_{j=0}^{k-1} \Psi_k diag\{\gamma(s-k, s-j)\}\Psi_j'.$$

Next, we consider

$$Var(\tilde{H}_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s} \Psi_k Var(\tilde{h}_{t+s-k}|\mathcal{F}_{t-1})\Psi_k' + 2\sum_{k=1}^{s}\sum_{j=0}^{k-1} \Psi_k Cov(\tilde{h}_{t+s-k}, \tilde{h}_{t+s-j}|\mathcal{F}_{t-1})\Psi_j'.$$

By Lemma 2(5) and 2(6), we can show that

$$Var(\tilde{h}_{t+s-k}|\mathcal{F}_{t-1}) = diag\{\tilde{v}_{s-k}\}.$$

Finally, we turn to the last term, $Cov(\tilde{h}_{t+s-k}, \tilde{h}_{t+s-j}|\mathcal{F}_{t-1})$. It can be also shown by Lemma 2(7) and 2(8) that for $j < k$,

$$Cov(\tilde{h}_{t+s-k}, \tilde{h}_{t+s-j}|\mathcal{F}_{t-1}) = diag\{\tilde{\gamma}(s-k, s-j)\}.$$

Combining all the results above, we have the desired result

$$\begin{aligned}\Delta_2(s) &= \sum_{k=0}^{s} \Psi_k diag\{\tilde{v}_{s-k} - v_{s-k}\}\Psi_k' \\ &\quad + 2\sum_{k=1}^{s}\sum_{j=0}^{k-1} \Psi_k diag\{\tilde{\gamma}(s-k, s-j) - \gamma(s-k, s-j)\}\Psi_j'.\end{aligned} \tag{29}$$

In order to complete the proof, we need to show that the covariance terms in (25) are zero; that is, $Cov(\tilde{E}_{t,s}, \tilde{H}_{t,s}|\mathcal{F}_{t-1}) = Cov(E_{t,s}, H_{t,s}|\mathcal{F}_{t-1}) = 0$. Consider

$$Cov(E_{t,s}, H_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s}\sum_{j=0}^{s} \Phi_k Cov(\varepsilon_{t+s-k}, h_{t+s-k}|\mathcal{F}_{t-1})\Psi_k' = 0$$

by Lemma 2(9), and

$$Cov(\tilde{E}_{t,s}, \tilde{H}_{t,s}|\mathcal{F}_{t-1}) = \sum_{k=0}^{s}\sum_{j=0}^{s} \Phi_k Cov(\tilde{\varepsilon}_{t+s-k}, \tilde{h}_{t+s-k}|\mathcal{F}_{t-1})\Psi_k' = 0$$

by Lemma 2(10). Therefore, we finally obtain the VIRF as desired:

$$\Delta_V(s) = \Delta_1(s) + \Delta_2(s),$$

where $\Delta_1(s)$ and $\Delta_2(s)$ are given in (28) and (29), respectively. ■

Appendix B

This appendix explains how the variance stability condition, $\omega\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 < 1$, is derived. To do so, we use the simplest univariate model as follows:

$$\begin{aligned} y_t &= \Lambda h_t + \varepsilon_t, \\ \varepsilon_t &= h_t^{1/2}\eta_t, \\ \eta_t &\sim IIN(0,1), \\ h_t &= \beta_0 + \beta_1\varepsilon_{t-1}^2 + \beta_2 h_{t-1}, \end{aligned}$$

where $h_t = Var(\varepsilon_t|\mathcal{F}_{t-1})$, and all the variables and parameters above are univariate scalars. We first explain how the familiar mean stability condition, $\beta_1 + \beta_2 < 1$, is derived. To compute the MIRF, we must evaluate terms such as $E(y_{t+s}|\mathcal{F}_{t-1})$. Consider the following:

$$y_{t+s} = \Lambda h_{t+s} + h_{t+s}^{1/2}\eta_{t+s}, \tag{30}$$

which implies that

$$E(y_{t+s}|\mathcal{F}_{t-1}) = \Lambda E(h_{t+s}|\mathcal{F}_{t-1}).$$

It can be shown that

$$E(h_{t+s}|\mathcal{F}_{t-1}) = \beta_0 + (\beta_1 + \beta_2)E(h_{t+s-1}|\mathcal{F}_{t-1}),$$

which is a first-order difference equation. Therefore, $E(y_{t+s}|\mathcal{F}_{t-1})$ is convergent only when $\beta_1 + \beta_2 < 1$ as forecast horizon s goes to infinity. To ensure the stability of the conditional mean function, $E(y_{t+s}|\mathcal{F}_{t-1})$, mean stability condition $\beta_1 + \beta_2 < 1$ should be in place.

Next we turn to the variance stability condition. Using the recursion $h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1}$, it is easily shown that

$$E(h_{t+s}|\mathcal{F}_{t-1}) = \beta_0 \sum_{k=0}^{s-1} (\beta_1 + \beta_2)^k + (\beta_1 + \beta_2)^s h_t. \tag{31}$$

To compute the VIRF, we need to calculate terms like $Var(y_{t+s}|\mathcal{F}_{t-1}) = E(y_{t+s}^2|\mathcal{F}_{t-1}) - [E(y_{t+s}|\mathcal{F}_{t-1})]^2$. To explain why the variance stability condition is needed, it is sufficient to consider $E(y_{t+s}^2|\mathcal{F}_{t-1})$ only. We obtain from (30)

$$E(y_{t+s}^2|\mathcal{F}_{t-1}) = \Lambda^2 E(h_{t+s}^2|\mathcal{F}_{t-1}) + E(h_{t+s}|\mathcal{F}_{t-1}) \tag{32}$$

since $E(h_{t+s}^{3/2}\eta_{t+s}|\mathcal{F}_{t-1}) = 0$ and $E(h_{t+s}\eta_{t+s}^2|\mathcal{F}_{t-1}) = E(h_{t+s}|\mathcal{F}_{t-1})$. The second term in (32) is convergent if $\beta_1 + \beta_2 < 1$, as shown above. Hence, it is sufficient to derive the condition under which the first term, $E(h_{t+s}^2|\mathcal{F}_{t-1})$, is convergent. Consider the following:

$$\begin{aligned} h_{t+s} &= \beta_0 + \beta_1 \varepsilon_{t+s-1}^2 + \beta_2 h_{t+s-1} \\ &= \beta_0 + \alpha_{t+s-1} h_{t+s-1}, \end{aligned}$$

where $\alpha_{t+s-1} = \beta_1 \eta_{t+s-1}^2 + \beta_2$. Therefore, we have

$$E(h_{t+s}^2|\mathcal{F}_{t-1}) = \beta_0^2 + 2\beta_0 E(\alpha_{t+s-1}h_{t+s-1}|\mathcal{F}_{t-1}) + E(\alpha_{t+s-1}^2 h_{t+s-1}^2|\mathcal{F}_{t-1}). \tag{33}$$

The second term in (33) is stable as long as the mean stability condition is satisfied because

$$E(\alpha_{t+s-1}h_{t+s-1}|\mathcal{F}_{t-1}) = \beta_0 \sum_{k=1}^{s-1} (\beta_1 + \beta_2)^k + (\beta_1 + \beta_2)^s h_t.$$

However, the third term in (33) is shown to have the following expression:

$$E(\alpha_{t+s-1}^2 h_{t+s-1}^2|\mathcal{F}_{t-1}) = (\omega\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2) E(h_{t+s-1}^2|\mathcal{F}_{t-1}).$$

Hence, combining the results above shows that $E(h_{t+s}^2|\mathcal{F}_{t-1})$ is in the form of a difference equation of order 1:

$$E(h_{t+s}^2|\mathcal{F}_{t-1}) = \mu_s + \kappa (h_{t+s-1}^2|\mathcal{F}_{t-1}) \tag{34}$$

where

$$
\begin{aligned}
\mu_s &= \beta_0^2 + 2\beta_0\beta_0 \sum_{k=1}^{s-1} (\beta_1 + \beta_2)^k + 2\beta_0(\beta_1 + \beta_2)^s h_t \\
\kappa &= \omega\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2.
\end{aligned}
$$

Note that the term μ_s is convergent if $\beta_1 + \beta_2 < 1$ as forecast horizon s goes to infinity. However, the solution for $E(h_{t+s}^2|\mathcal{F}_{t-1})$ in the first-order difference equation in (34) is stable (or convergent) only if $\kappa < 1$, which is the same as the variance stability condition. Therefore, $Var(y_{t+s}|\mathcal{F}_{t-1})$ is convergent only when $\omega\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 < 1$.

The reason the variance stability condition is needed is that conditional volatility term h_t enters the mean equation directly. We can show that if conditional standard deviation term $h_t^{1/2}$ enters the mean equation, as done in many empirical papers, then the variance stability condition is not required. In such a model, the usual mean stability condition, $\beta_1 + \beta_2 < 1$, is sufficient. To demonstrate, we consider the following simple model:

$$
\begin{aligned}
y_t &= \Lambda h_t^{1/2} + \varepsilon_t, \\
\varepsilon_t &= h_t^{1/2}\eta_t, \\
\eta_t &\sim IIN(0,1), \\
h_t &= \beta_0 + \beta_1\varepsilon_{t-1}^2 + \beta_2 h_{t-1}.
\end{aligned}
$$

It is the exactly same model as before, but $h_t^{1/2}$ appears instead of h_t. Now, we consider

$$
y_{t+s} = \Lambda h_{t+s}^{1/2} + h_{t+s}^{1/2}\eta_{t+s},
$$

which implies that

$$
E(y_{t+s}|\mathcal{F}_{t-1}) = \Lambda E(h_{t+s}^{1/2}|\mathcal{F}_{t-1}).
$$

Since the square root function is concave, we have $E(y_{t+s}|\mathcal{F}_{t-1}) \leq [E(h_{t+s}|\mathcal{F}_{t-1})]^{1/2}$. Hence, $\beta_1+\beta_2 < 1$ will ensure that $E(h_{t+s}|\mathcal{F}_{t-1})$ is stable, which in turn implies that $E(y_{t+s}|\mathcal{F}_{t-1})$ is stable as s goes to infinity. Next, we turn to $Var(y_{t+s}|\mathcal{F}_{t-1}) = E(y_{t+s}^2|\mathcal{F}_{t-1}) - [E(y_{t+s}|\mathcal{F}_{t-1})]^2$. Again, the key term is $E(y_{t+s}^2|\mathcal{F}_{t-1})$, which is given by

$$
\begin{aligned}
E(y_{t+s}^2|\mathcal{F}_{t-1}) &= \Lambda^2 E(h_{t+s}|\mathcal{F}_{t-1}) + 2\Lambda E(h_{t+s}\eta_{t+s}|\mathcal{F}_{t-1}) + E(h_{t+s}\eta_{t+s}^2|\mathcal{F}_{t-1}) \\
&= (\Lambda^2 + 1)E(h_{t+s}|\mathcal{F}_{t-1}).
\end{aligned}
$$

Unlike the previous case, $E(y_{t+s}^2|\mathcal{F}_{t-1})$ is a function of $E(h_{t+s}|\mathcal{F}_{t-1})$, not of $E(h_{t+s}^2|\mathcal{F}_{t-1})$.

Therefore, mean stability condition $\beta_1+\beta_2 < 1$ will guarantee that all the terms, $E(h_{t+s}|\mathcal{F}_{t-1})$, $E(y^2_{t+s}|\mathcal{F}_{t-1})$, and $Var(y_{t+s}|\mathcal{F}_{t-1})$, are jointly stable.

Table 1. Summary statistics based on 183 observations (sample period: 1974:Q2 to 2019:Q4)

	Sample mean	Sample standard deviation	Max	Min	Sample correlation
Growth Rate of Real Oil Price	1.08%	55.89%	142.62%	-291.59%	0.1289
Growth Rate of Real GDP	2.67%	3.00%	15.17%	-8.83%	

Table 2. ML estimates for the coefficients of GARCH(1,1) equations

	Constant (β_0)	ARCH term (β_1)	GARCH term (β_2)
Oil GARCH equation (h_{1t})	468.76 (83.85)	0.34 (3.13)	0.53 (7.75)
GDP GARCH equation (h_{2t})	0.75 (2.30)	0.18 (2.23)	0.70 (9.11)

Note: t-statistics are in parentheses.

Table 3. Mean-stability and variance-stability conditions

	Mean-stability condition $\beta_1 + \beta_2 < 1$	Variance-stability condition $3\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 < 1$
Oil GARCH equation	0.869	0.987
GDP GARCH equation	0.907	0.892

Table 4. ML estimates for the first AR coefficient matrix A_1

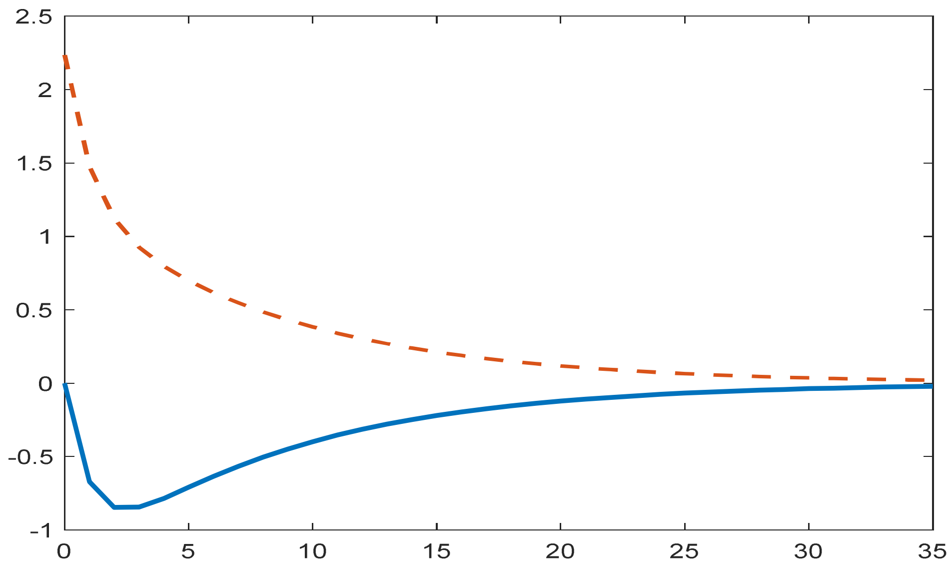
	First lag of oil	First lag of GDP
Oil mean equation	0.3212 (3.1694)	0.9637 (0.8917)
GDP mean equation	-0.0069 (-1.9302)	0.2151 (2.4405)

Note: t-statistics are in parentheses.

Table 5. ML estimates for coefficients Λ of GARCH-in-Mean terms

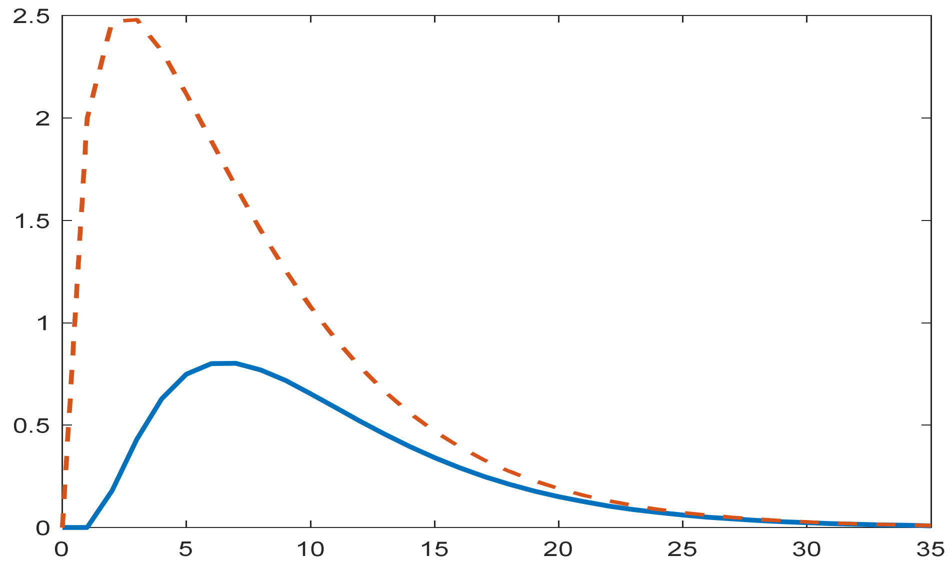
	Oil volatility h_{1t}	GDP volatility h_{2t}
Oil mean equation	0.0026 (0.9057)	-0.3227 (-0.3959)
GDP mean equation	-0.0002 (-2.2340)	-0.0637 (-0.6599)

Note: t-statistics are in parentheses.



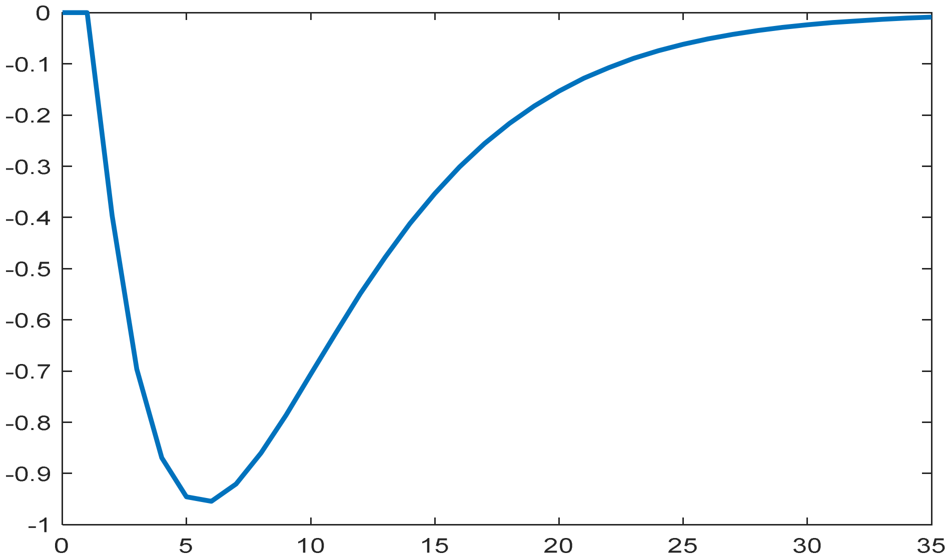
Note: The MIRF of y_{1t} is indicated by the solid line, and the MIRF of y_{2t} is the dashed line. There is no GARCH-in-Mean effect in the model. A mean shift impulse is applied to y_{2t} with $\delta_2 = 2.24$.

Figure 1. Mean Impulse Response Functions (Level Shock)



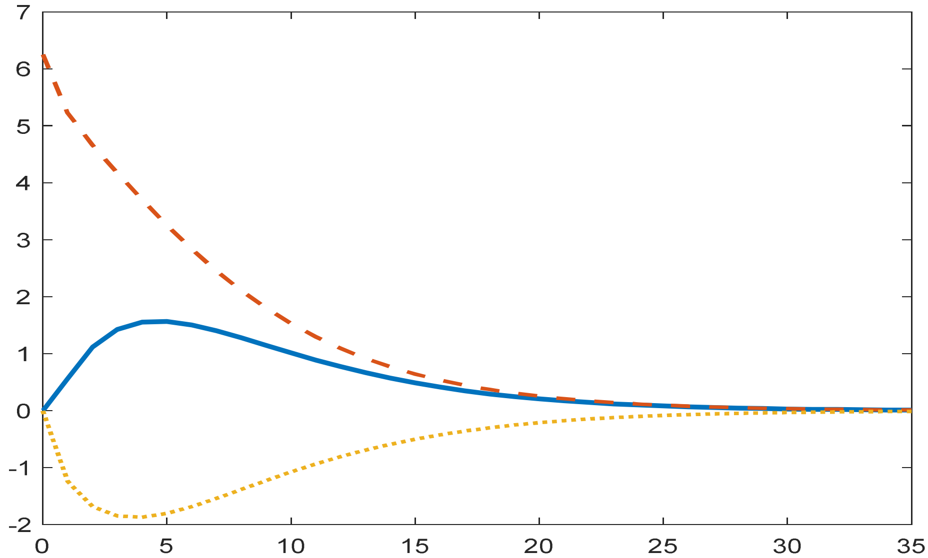
Note: The VIRF of y_{1t} is indicated by the solid line, and the VIRF of y_{2t} is the dashed line. There is no GARCH-in-Mean effect in the model. A mean shift impulse is applied to y_{2t} with $\delta_2 = 2.24$.

Figure 2. Variance Impulse Response Functions (Level Shock)



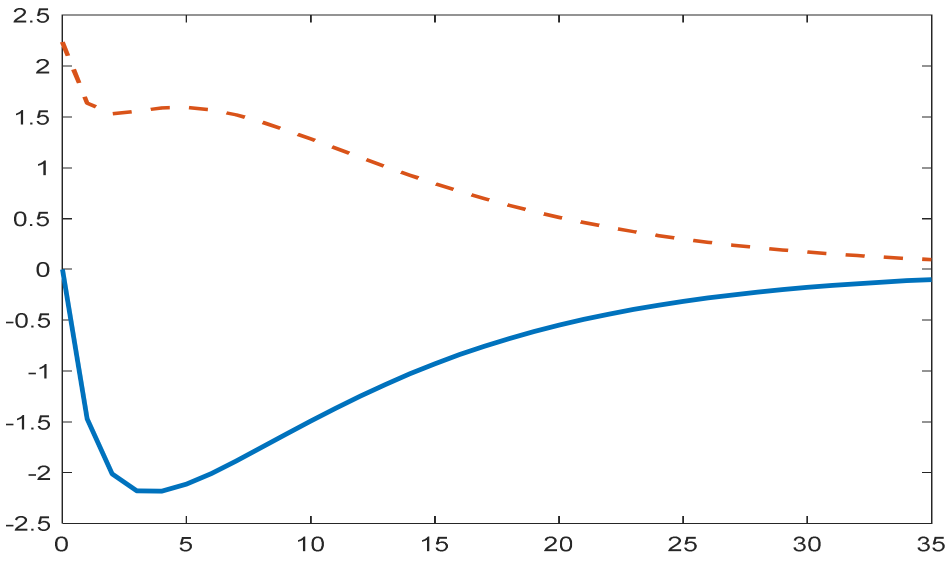
Note: The CIRF between y_{1t} and y_{2t} is indicated by the solid line. There is no GARCH-in-Mean effect in the model. A mean shift impulse is applied to y_{2t} with $\delta_2 = 2.24$.

Figure 3. Covariance Impulse Response Function (Level Shock)



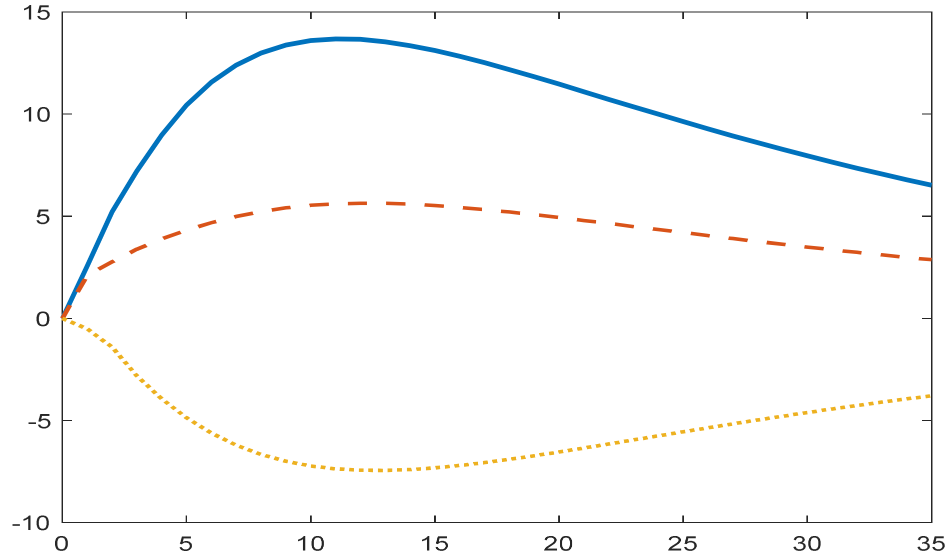
Note: The VIRF of y_{1t} is indicated by the solid line, and the VIRF of y_{2t} is the dashed line. The CIRF between y_{1t} and y_{2t} is the dotted line. The model is identical to the one used in Figures 2 & 3. However, a scale shift impulse is applied to y_{2t} with $\lambda_2 = 1.5$.

Figure 4. Variance & Covariance Impulse Response Functions (Uncertainty Shock)



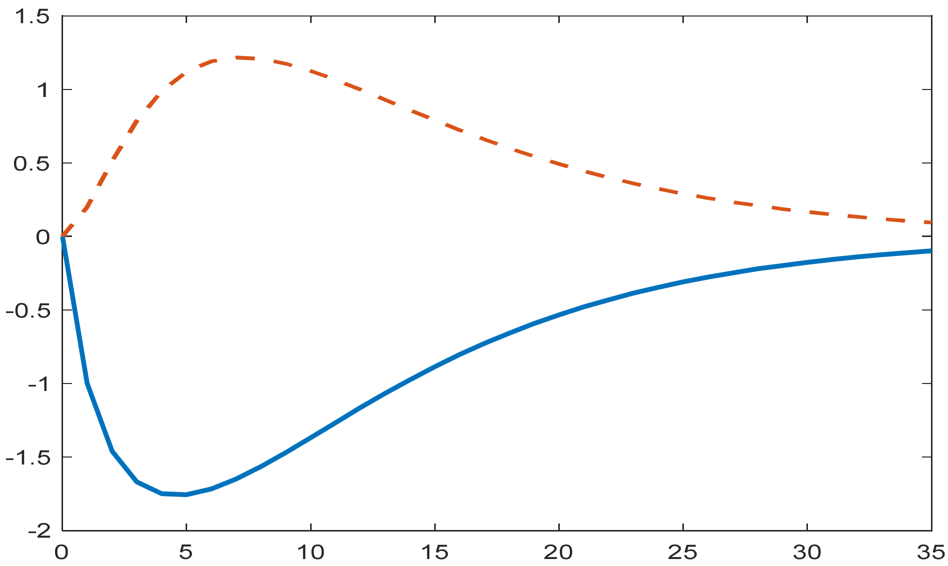
Note: The MIRF of y_{1t} is indicated by the solid line, and the MIRF of y_{2t} is the dashed line. The model specification and impulse scheme are the same as in Figure 1 except for the presence of the GARCH-in-Mean effect.

Figure 5. Mean Impulse Response Functions (Level Shock with GARCH-in-Mean)



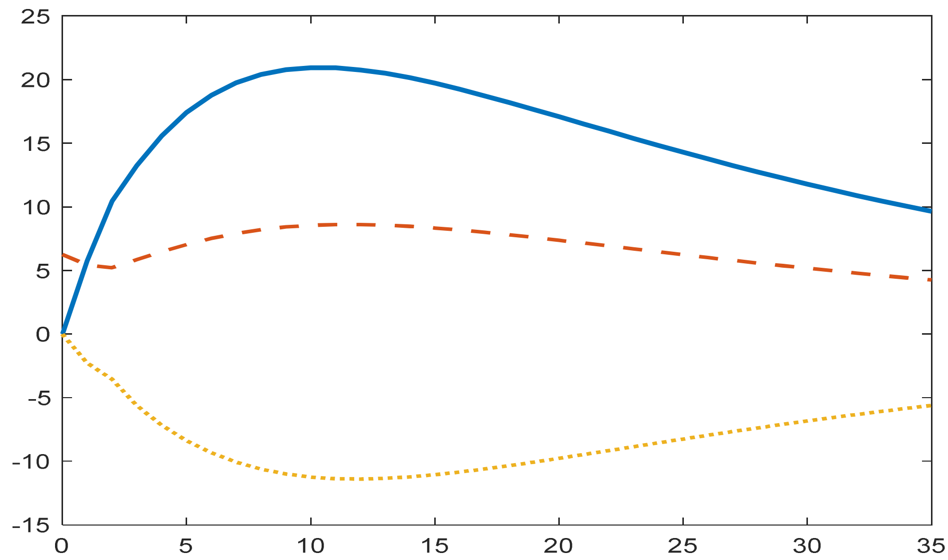
Note: The VIRF of y_{1t} is indicated by the solid line, and the VIRF of y_{2t} is the dashed line. The CIRF between y_{1t} and y_{2t} is the dotted line. The model specification and impulse scheme are the same as in Figure 1 except for the presence of the GARCH-in-Mean effect.

Figure 6. Variance & Covariance Impulse Response Functions (Level Shock with GARCH-in-Mean)



Note: The MIRF of y_{1t} is indicated by the solid line, and the MIRF of y_{2t} is the dashed line. The model specification and impulse scheme are the same as in Figure 4 except for the presence of the GARCH-in-Mean effect.

Figure 7. Mean Impulse Response Functions (Uncertainty Shock with GARCH-in-Mean)



Note: The VIRF of y_{1t} is indicated by the solid line, and the VIRF of y_{2t} is the dashed line. The CIRF between y_{1t} and y_{2t} is the dotted line. The model specification and impulse scheme are the same as in Figure 4 except for the presence of the GARCH-in-Mean effect.

Figure 8. Variance & Covariance Impulse Response Functions (Uncertainty Shock with GARCH-in-Mean)

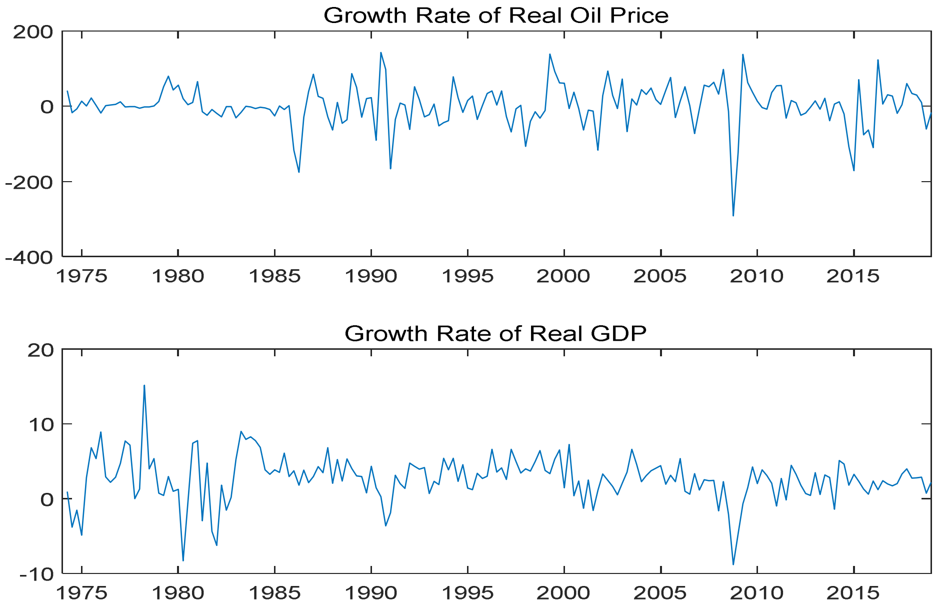
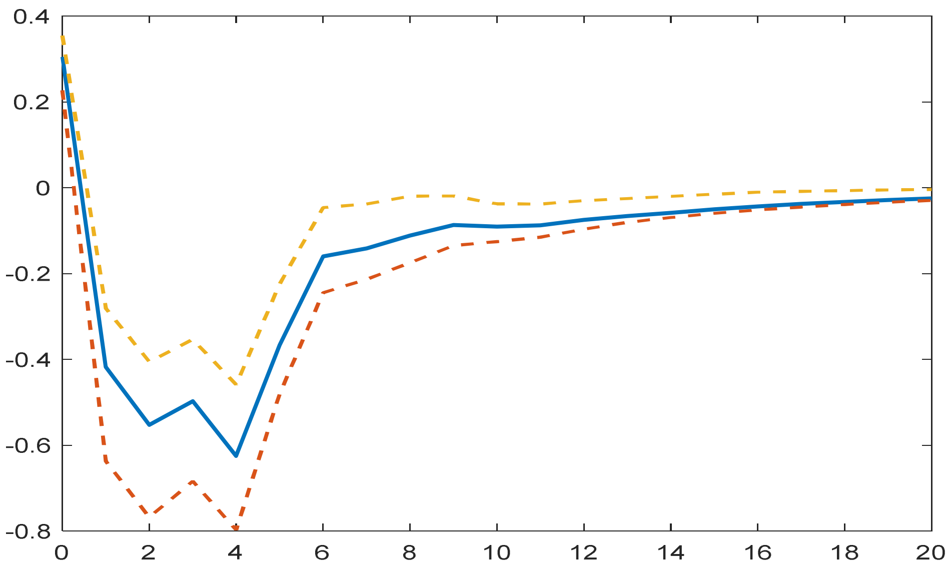
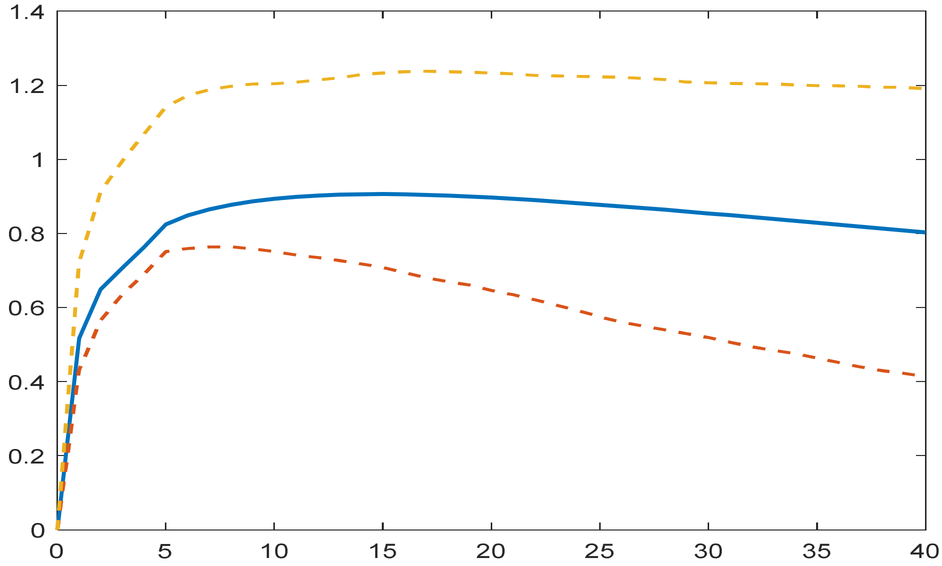


Figure 9. Time-series Plots of GDP and Oil Price



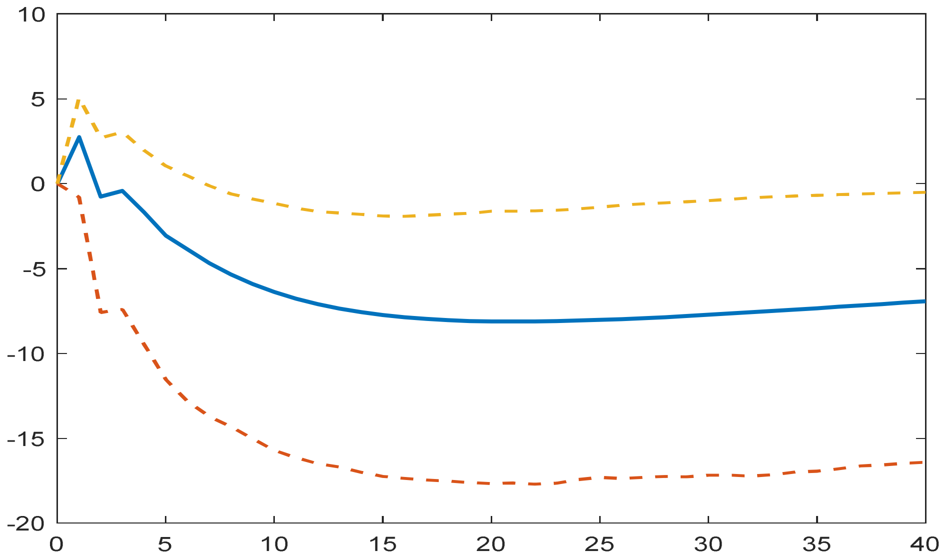
Note: The MIRF of y_{2t} (GDP growth) is indicated by the solid line with one-standard deviation bootstrap confidence intervals (dashed lines). The units for the horizontal and vertical axes are the number of quarters and the percentage, respectively. The intervention is applied only to the location of the structural shock of y_{1t}, and the size of the intervention is one standard deviation of the corresponding error term (estimated value at 53.24%).

Figure 10. Mean Impulse Response Function of Output (Level Shock)



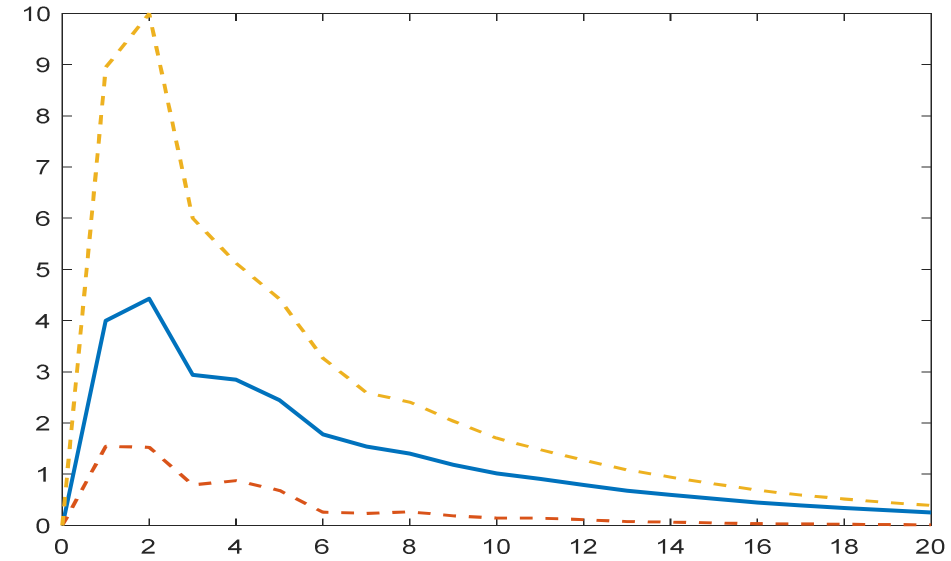
Note: The VIRF of y_{2t} (GDP growth) is indicated by the solid line with one-standard deviation bootstrap confidence intervals (dashed lines). The units for the horizontal and vertical axes are the number of quarters and the squared percentage, respectively. The intervention is applied only to the location of the structural shock of y_{1t}, and the size of the intervention is one standard deviation of the corresponding error term, (estimated value at 53.24%).

Figure 11. Variance Impulse Response Function of Output (Level Shock)



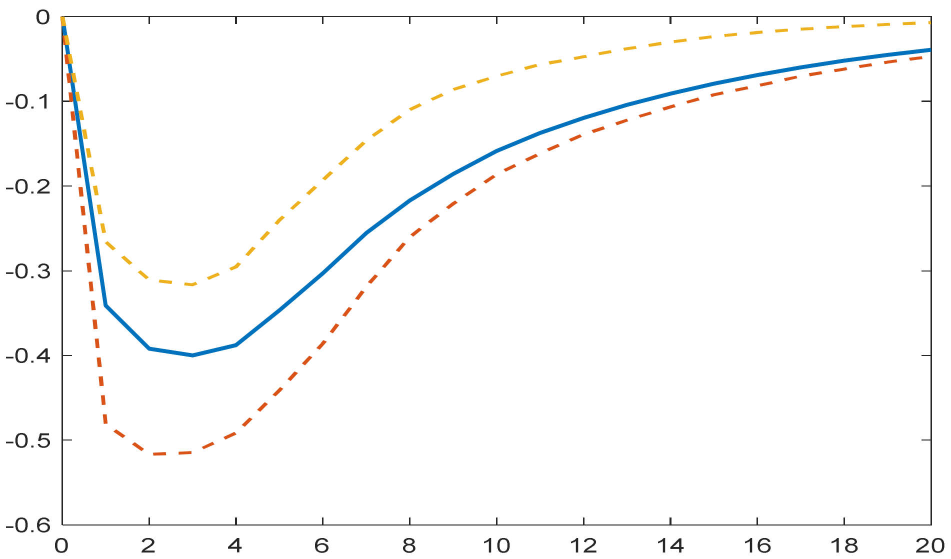
Note: The CIRF of y_{1t} (oil prices) and y_{2t} (GDP growth) is indicated by the solid line with one-standard deviation bootstrap confidence intervals (dashed lines). The units for the horizontal and vertical axes are the number of quarters and the squared percentage, respectively. The intervention is applied only to the location of the structural shock of y_{1t}, and the size of the intervention is one standard deviation of the corresponding error term, (estimated value at 53.24%).

Figure 12. Covariance Impulse Response Function of Oil Prices and Output (Level Shock)



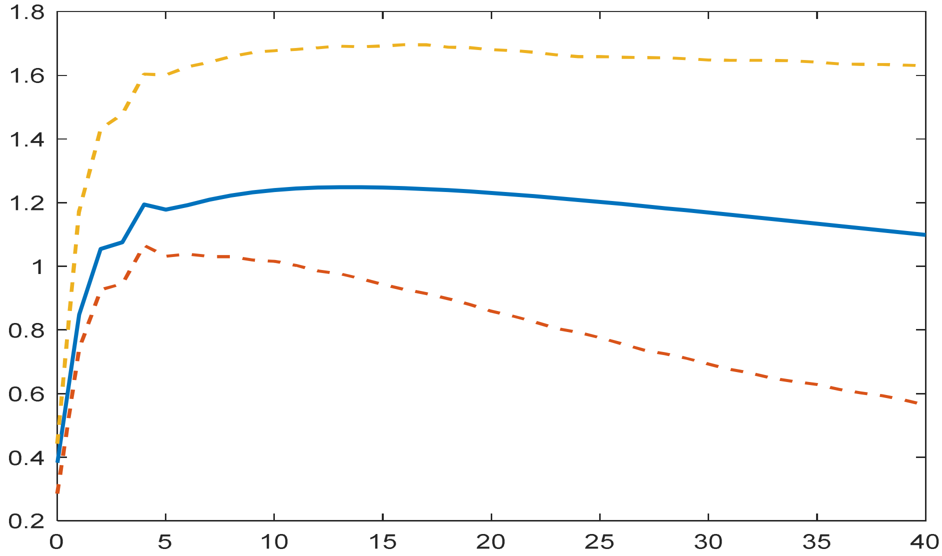
Note: The MIRF of y_{1t} (oil prices) is indicated by the solid line with one-standard deviation bootstrap confidence intervals (dashed lines). The units for the horizontal and vertical axes are the number of quarters and the percentage, respectively. The intervention is applied only to the uncertainty of the structural shock of y_{1t}, and the size of the intervention is 50%.

Figure 13. Mean Impulse Response Function of Oil Prices (Uncertainty Shock)



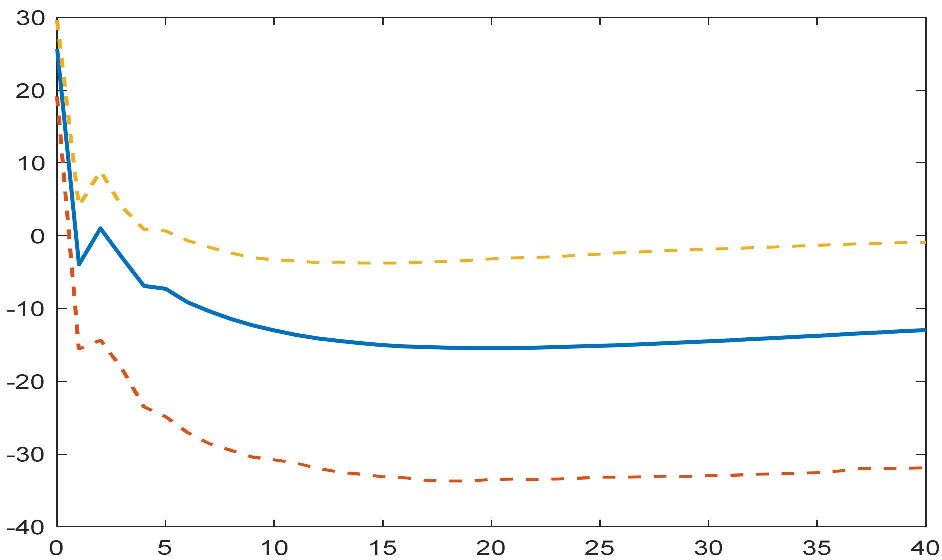
Note: The MIRF of y_{2t} (GDP growth) is indicated by the solid line with one-standard deviation bootstrap confidence intervals (dashed lines). The units for the horizontal and vertical axes are the number of quarters and the percentage, respectively. The intervention is applied only to the uncertainty of the structural shock to y_{1t}, and the size of the intervention is 50%.

Figure 14. Mean Impulse Response Function of Output (Uncertainty Shock)



Note: The VIRF of y_{2t} (GDP growth) is indicated by the solid line with one-standard deviation bootstrap confidence intervals (dashed lines). The units for the horizontal and vertical axes are the number of quarters and the squared percentage, respectively. The intervention is applied only to the uncertainty of the structural shock to y_{1t}, and the size of the intervention is 50%.

Figure 15. Variance Impulse Response Function of Output (Uncertainty Shock)



Note: The CIRF of y_{1t} (oil prices) and y_{2t} (GDP growth) is indicated by the solid line with one-standard deviation bootstrap confidence intervals (dashed lines). The units for the horizontal and vertical axes are the number of quarters and the squared percentage, respectively. The intervention is applied only to the uncertainty of the structural shock to y_{1t}, and the size of the intervention is 50%.

Figure 16. Covariance Impulse Response Function of Oil Prices and Output (Uncertainty Shock)